Paper 1 Jeopardy [145 marks]

1. [Maximum mark: 5] 20N.1.AHL.TZ0.H_5

The first term in an arithmetic sequence is 4 and the fifth term is $\log_2 625$.

Find the common difference of the sequence, expressing your answer in the form $\log_2 p$, where $p \in \mathbb{Q}$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2\left(\frac{625}{16}\right)$

$d = \frac{1}{4}\log_2\left(\frac{625}{16}\right)$

attempt to use power rule for logs ***M1***

$d = \log_2\left(\frac{625}{16}\right)^{\frac{1}{4}}$

$d = \log_2\left(\frac{5}{2}\right)$ ***A1***

[5 marks]

Note: Award method marks in any order.

2. [Maximum mark: 6] 25N.1.AHL.TZ1.6

A linear system of equations is given by

$x - z = 4$

$2x + 2y + z = 14$

$x + 2y + \alpha z = \beta$

Find the value of α and the value of β, where $\alpha, \beta \in \mathbb{Z}$, for which the system has an infinite number of solutions.

[6]

Markscheme

METHOD 1 Row Reduction

attempt to reduce a matrix or eliminate one variable ***(M1)***

two correct rows or equivalent equations ***A1A1***

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 2 & 2 & 1 & 14 \\ 1 & 2 & \alpha & \beta \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 2 & 3 & 6 \\ 0 & 2 & 1+\alpha & \beta - 4 \end{bmatrix}$$

correct third row or equivalent equation ***A1***

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & \alpha - 2 & \beta - 10 \end{bmatrix}$$

equating the third row coefficients to zero ***(M1)***

$\alpha = 2$ and $\beta = 10$ ***A1***

METHOD 2 Writing an equation in one variable

attempt to eliminate either x or z using the first two equations (seen anywhere) ***(M1)***

$2(z+4) + 2y + z = 14$ **OR** $2x + 2y + (x-4) = 14$

one correct equation *A1*

$2y + 3z = 6$ **OR** $3x + 2y = 18$

correct substitution into the third equation to create an equation in either x or z *A2*

$x + 18 - 3x + \alpha(x - 4) = \beta$ **OR** $z + 4 + 6 - 3z + \alpha z = \beta$ **OR** $(\alpha - 2)x = \beta + 4\alpha - 18$ **OR** $z(\alpha - 2) = \beta - 10$

attempt to set coefficients equal to zero for infinite solutions *(M1)*

$\alpha = 2$ and $\beta = 10$ *A1*

METHOD 3 Subtracting and comparing equations

attempt to add or subtract two equations *(M1)*

$x + 2y + 2z = 10$ **OR** $2x + 2y + (\alpha - 1)z = \beta + 4$ **OR** $x + (1 - \alpha)z = 14 - \beta$ *A2*

attempt to compare coefficients with the third equation *(M1)*

$\alpha - 1 = 1$, $\beta + 4 = 14$, $1 - \alpha = -1$, $14 - \beta = 4$

$\alpha = 2$ and $\beta = 10$ *A1A1*

METHOD 4 Vector approach

attempt to find direction vector of line of intersection using Π_1 and Π_2 *(M1)*

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\left(\vec{d} =\right) \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$ *A1*

normal of Π_3 is perpendicular to $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ ***(M1)***

$$\begin{pmatrix} 1 \\ 2 \\ \alpha \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = 0$$

$\alpha = 2$ ***A1***

correct substitution of a point common to the first two planes into the third plane ***A1***

$0 + 2(9) + 2(-4)$, $4 + 2(3) + 2(0)$

$\beta = 10$ ***A1***

METHOD 5 Finding a general point on the line of intersection

attempt to find intersection of Π_1 and Π_2 ***(M1)***
$x = 4 + z$, $y = 3 - \frac{3}{2}z$

$x = 4 + 2t$, $y = 3 - 3t$, $z = 2t$ **OR** $\mathbf{r} = (4,\ 3,\ 0) + t(2,\ -3,\ 2)$ ***A1A1***

correct substitution of general point into Π_3 ***A1***

$(4 + 2t) + 2(3 - 3t) + \alpha(2t)$, $10 + t(2\alpha - 4)$

recognizing that the coefficient of t must be zero **OR** the constant term must be β ***(M1)***

$2\alpha - 4 = 0$ **OR** $\beta = 10$

$\alpha = 2$ and $\beta = 10$ ***A1***

[6 marks]

Examiners report

This proved to be quite a challenging question for many students, particularly if they attempted methods other than a matrix reduction. Infinite solutions representing the coefficients of their final equations being zero was often missed, leading to a maximum of four marks. A fair number of students attempted, and were often successful, solving this problem using vector methods.

3. [Maximum mark: 5] 25N.1.AHL.TZ3.4

Solve the equation $3\log_8 10x - \log_4 x = 1$ for $x > 0$. [5]

Markscheme

Note: Candidates may approach this problem in different ways and may do their steps in many different orders.
The ***M*** marks are independent and may be awarded in any order. Use the description of each mark to determine when the mark may be awarded.

$$3\log_8 10x - \log_4 x = 1$$

attempt to apply power rule, quotient rule, or product rule (seen anywhere) ***(M1)***

attempt to apply change of base rule (seen anywhere) ***(M1)***

correct equation with same base in all logarithms ***A1***

$\log_2 10x - \log_2 x^{\frac{1}{2}} = 1$ or $\log_2 10 + \log_2 x - \frac{1}{2}\log_2 x = 1$ or $\frac{\log_4 (10x)^3}{\log_4 8} - \log_4 x = 1$ (or equivalent)

correct equation without logarithms ***A1***

$10\sqrt{x} = 2$ or $\sqrt{x} = \frac{2}{10}\left(= \frac{1}{5}\right)$ (or equivalent)

$x = \frac{1}{25}$ ***A1***

[5 marks]

Examiners report

SL:

Many candidates struggled with this question, which required them to solve an equation containing logarithms with different bases. That said, a good number of candidates were able to earn one or two method marks for their attempts to apply the change of base rule or the power rule for logarithms. A commonly seen incorrect attempt to apply the rules of logarithms was to express $3\ \log_8\ 10x$ as $\log_8\ 10x^3$, neglecting to apply the exponent to both the 10 and the x. While some were able to correctly apply the change of base rule and create a correct equation in terms of base 2, 4 or 8, only a small percentage of candidates were able to earn full marks here due to various algebraic errors in their working.

HL:

The range of responses to this question seemed significant in terms of the abilities of students attempting this paper, as attempts ranged from poor applications of the 'power rule' to fully correct answers seen within three or four lines of working. Regarding the former, it was not unusual to see $3\ \log\ 10x$ written as $\log\ 10x^3$, rather than
$\log\ (10x)^3$, which caused issues with subsequent working. The best, and most perceptive answers, jumped straight to stating $\log_2\ 10x - \frac{1}{2}\log_2\ x = 1$, from where it was relatively straightforward to reach the correct answer of $x = \frac{1}{25}$.

4. [Maximum mark: 7] 24M.1.AHL.TZ2.9

A teacher takes n students on a field trip. The students are assigned randomly into two groups.

For safety reasons there must be exactly three students in the first group and at least three students in the second group.

The teacher will randomly assign three students to the first group and the other students to the second group.

(a) Write down an expression for the number of ways that the students could be assigned. [1]

Markscheme

${}^{n}C_{3}$ ***A1***

[1 mark]

Examiners report

This proved to be a challenging question so it was nice to see students achieving part marks on the way with thoughtful application. Part (a) was often correct although a common error was not to recognise that choosing three students to be in the first group made the second group choice automatic. In part (b) it was pleasing to see candidates gaining marks by showing clear working, setting up relevant equations and simplifying their factorials.

Two of the students ask the teacher not to work in the same group.

The teacher agrees and now finds that the number of ways to assign the students is halved.

(b) Determine the value of n. [6]

Markscheme

EITHER

finding the number of ways to assign the students with the two students apart

number of ways to assign two students ${}^{2}C_{1}$ (seen anywhere) ***(A1)***

number of ways to assign others ${}^{n-2}C_{2}$ to have one group of 3 (seen anywhere) ***(A1)***

number of ways $= {}^{2}C_{1} \times {}^{n-2}C_{2}$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}{}^{n}C_{3} = {}^{2}C_{1} \times {}^{n-2}C_{2}$ OR ${}^{n}C_{3} = 2 \times {}^{2}C_{1} \times {}^{n-2}C_{2}$

valid attempt to eliminate all factorials from their equation ***(M1)***

$\frac{n(n-1)(n-2)}{3\times2} = 2 \times 2 \times \frac{(n-2)(n-3)}{2}$ or equivalent with no factorials

$n(n-1) = 12(n-3)$

OR

finding the number of ways to assign the students with the two students together

number of ways to assign two students and one other to the first group $^{n-2}C_1$ (seen anywhere) ***(A1)***

number of ways to assign three other students the first group $^{n-2}C_3$ (seen anywhere) ***(A1)***

number of ways $= {}^{n-2}C_1 + {}^{n-2}C_3$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}{}^{n}C_3 = {}^{n-2}C_1 + {}^{n-2}C_3$ OR ${}^{n}C_3 = 2\left({}^{n-2}C_1 + {}^{n-2}C_3\right)$

valid attempt to eliminate all factorials from their equation ***(M1)***

$\frac{n(n-1)(n-2)}{3\times2} = 2 \times (n-2) + 2 \times \frac{(n-2)(n-3)(n-4)}{3\times2}$

$n(n-1) = 12 + 2(n-3)(n-4)$

THEN

$n^2 - 13n + 36 = 0$ ***A1***

$(n-9)(n-4) = 0$

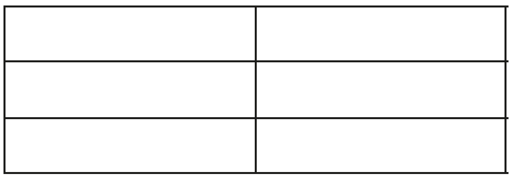
$n = 9$ ***A1***

Note: Do not award the final ***A1*** if additional values of n are given.

[6 marks]

5. [Maximum mark: 8] 21M.1.AHL.TZ1.9

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

Markscheme

METHOD 1

B has one less pen to select ***(M1)***

EITHER

A and B can be placed in 6×5 ways ***(A1)***

C, D, E have 6 choices each ***(A1)***

OR

A (or B), C, D, E have 6 choices each ***(A1)***

B (or A) has only 5 choices ***(A1)***

THEN

$5 \times 6^4 \; (= 6480)$ ***A1***

METHOD 2

total number of ways $= 6^5$ ***(A1)***

number of ways with Amber and Brownie together $= 6^4$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

$6^5 - 6^4$

$= 5 \times 6^4 \; (= 6480)$ ***A1***

[4 marks]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

Markscheme

METHOD 1

total number of ways $= 6!(= 720)$ ***(A1)***

number of ways with Amber and Brownie sharing a boundary

$= 2 \times 7 \times 4!(= 336)$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

$720 - 336 = 384$ ***A1***

METHOD 2

case 1: number of ways of placing A in corner pen

$3 \times 4 \times 3 \times 2 \times 1$

Four corners total no of ways is
$4 \times (3 \times 4 \times 3 \times 2 \times 1) = 12 \times 4!(= 288)$ ***(A1)***

case 2: number of ways of placing A in the middle pen

$2 \times 4 \times 3 \times 2 \times 1$

two middle pens so $2 \times (2 \times 4 \times 3 \times 2 \times 1) = 4 \times 4!(= 96)$ ***(A1)***

attempt to add (may be seen in words) ***(M1)***

total no of ways $= 288 + 96$

$= 16 \times 4!(= 384)$ ***A1***

[4 marks]

6. [Maximum mark: 5] 24M.1.AHL.TZ2.2

Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$. [5]

Markscheme

recognising a quadratic in 3^x ***(M1)***

$3 \times (3^x)^2 + 5 \times 3^x - 2 = 0$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) ***(M1)***

$(3 \times 3^x - 1)(3^x + 2) = 0$ OR $3^x = \frac{-5 \pm \sqrt{25+24}}{6}$ (or equivalent) ***(A1)***

$3^x = \frac{1}{3}$ (or $3^x = -2$) ***(A1)***

$x = -1$ ***A1***

Note: Award the final ***A1*** if candidate's answer includes $x = -1$ and $x = \log_3 (-2)$. Award ***A0*** if other incorrect answers are given.

[5 marks]

Examiners report

Many students recognised the hidden three term quadratic and as a consequence produced a correct solution. A number of students who did not recognise the quadratic tried to apply logarithms to the sum of terms.

7. [Maximum mark: 5] 25M.1.AHL.TZ3.3

Solve the equation $2 \cos 2\theta - 5 \cos \theta + 2 = 0$, where $\pi \leq \theta \leq 2\pi$. [5]

Markscheme

recognizing to use $\cos 2\theta = 2 \cos^2 \theta - 1$ ***(M1)***

$2\left(2 \cos^2 \theta - 1\right) - 5 \cos \theta + 2\ (= 0)$ ***A1***

$4 \cos^2 \theta - 5 \cos \theta\ \ (= 0)$

choosing an appropriate method to solve their quadratic equation ***(M1)***

$\cos \theta(4 \cos \theta - 5)$ **OR** $\frac{5 \pm \sqrt{(-5)^2 - 4 \times 4 \times 0}}{2 \times 4}$ ***(A1)***

$\cos \theta = 0$

$\theta = \frac{3\pi}{2}$ ***A1***

Note: Do not award final ***A1*** if any extra solutions given.

[5 marks]

Examiners report

There were a significant number of good solutions to this question, although there were a number of students that did not pay attention to the range of values for the solutions, and others who included an extra solution of $\cos^{-1}\frac{5}{4}$. A significant number of students approached the quadratic equation found, by dividing both sides by $\cos\theta$, resulting in a loss of most of the marks.

8. [Maximum mark: 5] 23N.1.AHL.TZ1.1

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

Markscheme

attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \ \left(= x^2 - 6x + 9 + k^2\right)$ ***A1***

[2 marks]

Examiners report

SL:

While most candidates answered part (a) correctly, some mistakenly found $(f \circ g)(x)$ rather than $(g \circ f)(x)$. In part (b), the most common error was to go from $k^2 = 9$ to a final answer of $k = 3$, neglecting the $\pm$.

HL:

A very straightforward question for almost all candidates. One or two arithmetical slips rarely occurred in part (b) where $k = \pm\sqrt{11}$ was sometimes seen.

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

Markscheme

substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2 - 3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

9. [Maximum mark: 5] 25M.1.AHL.TZ2.7

It is given that $x^4 + px^3 - 2x^2 + qx - 3$ is exactly divisible by $(x + 1)^2$.

Find the value of p and the value of q, where $p,\ q \in \mathbb{R}$. [5]

Markscheme

METHOD 1 (product of quadratic factors)

attempt to write as product of two quadratic factors ***(M1)***

$x^4 + px^3 - 2x^2 + qx - 3 = (x^2 + 2x + 1)(ax^2 + bx + c)$

$= (x^2 + 2x + 1)(x^2 + bx - 3)$ **OR** $(x^2 + 2x + 1)(x^2 - 3)$ ***A1***

attempt to compare their coefficients ***(M1)***

$p = b + 2$ **OR** $q = b - 6$ **OR** $-2 = 2b + 1 - 3$

$b = 0$ ***(A1)***

$p = 2,\ q = -6$ ***A1***

METHOD 2 (double root)

let $f(x) = x^4 + px^3 - 2x^2 + qx - 3$

recognition that $f(-1) = 0$ ***(M1)***

$p + q = -4$ ***A1***

recognition that $f\prime(x) = 4x^3 + 3px^2 - 4x + q$

$3p + q = 0$ ***A1***

$p = 2,\ q = -6$ ***A1***

METHOD 3 (division by $(x + 1)$ twice)

attempt to use division with $(x + 1)$ to find remainder ***(M1)***

$p + q = -4$ ***A1***

attempt to use division once again with $(x + 1)$ to find remainder ***(M1)***

$q + 3p = 0$ ***A1***

$p = 2,\ q = -6$ ***A1***

METHOD 4 (division by $x^2 + 2x + 1$)

attempt to use division to find remainder ***(M1)***

$(q + 3p)x + 2p - 4(= 0)$ (or equivalent) ***(A1)***

equating the coefficients of their remainder to 0 ***(M1)***

$q + 3p = 0$ and $2p - 4 = 0$ **OR** $q - p + 2 = 2(1 - 2p)$ and $-3 = 1 - 2p$ (or eqivalent) ***A1***

$p = 2,\ q = -6$ ***A1***

METHOD 5 (sum and product of roots)

let the four roots be $-1,\ -1,\ \alpha$ and β

attempt to set sum of four roots equal to $-p$ **OR** product of four roots equal to -3 ***(M1)***

Note: Award ***M1*** for expansion of $(x-1)^2(x-\alpha)(x-\beta)$ leading to a quartic $x^4 + (2-\alpha-\beta)x^3 + (\alpha\beta - 2\alpha - 2\beta + 1)x^2 + (2\alpha\beta - \alpha - \beta)x + \alpha\beta(=0)$

$(-1) + (-1) + \alpha + \beta = -p(\Rightarrow \alpha + \beta = -p + 2)$ and $(-1)(-1)\alpha\beta = -3(\Rightarrow \alpha\beta = -3)$ ***A1***

sum of product of pairs and sum of product of roots taken three at a time

$\alpha\beta - \alpha - \alpha - \beta - \beta + 1 = -2(\Rightarrow \alpha\beta - 2\alpha - 2\beta = -3)$ and $-\alpha\beta - \alpha\beta + \beta + \alpha = -q(\Rightarrow -2\alpha\beta + \beta + \alpha = -q)$ ***A1***

Note: Award ***A1*** for any two correct equations.

attempt to solve *their* four equations to obtain the value of p and q ***(M1)***

$\alpha\beta = -3$ and $\alpha\beta - 2\alpha - 2\beta = -3 \Rightarrow \alpha + \beta = 0$

$p = 2,\ q = -6$ ***A1***

[5 marks]

Examiners report

Most candidates correctly applied the Factor Theorem to form an initial linear equation involving p and q, often earning the first two marks. A variety of methods was attempted, including the use of sum and product of roots, direct factorisation or algebraic division, with some candidates switching approaches when their first method was unsuccessful. However, many found it difficult to form a second correct equation. Stronger candidates were able to do so and produced fully correct solutions.

10. [Maximum mark: 7] 22N.1.AHL.TZ0.5

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i},\ \alpha$ and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

Markscheme

product of roots $= 80$ ***(A1)***

$3 - \mathrm{i}$ is a root ***(A1)***

attempt to set up an equation involving the product of their four roots and ± 80 ***(M1)***

$(3 + \mathrm{i})(3 - \mathrm{i})\alpha^3 = 80 \Rightarrow 10\alpha^3 = 80$

$\alpha = 2$ ***A1***

[4 marks]

Examiners report

This proved to be troublesome only for those who failed to recognise the existence of the conjugate root $3 - \mathrm{i}$ and proceeded to work with three roots rather than the required four. Otherwise, it was very pleasing to see a great number of candidates gain full marks in both question parts.

(b) Find the value of p. [3]

Markscheme

METHOD 1

sum of roots $= -p$ ***(A1)***

$-p = 3 + \mathrm{i} + 3 - \mathrm{i} + 2 + 4$ ***(M1)***

Note: Accept $p = 3 + \mathrm{i} + 3 - \mathrm{i} + 2 + 4$ for ***(M1)***

$p = -12$ ***A1***

METHOD 2

$(z - (3 + \mathrm{i}))(z - (3 - \mathrm{i}))(z - 2)(z - 4)$ ***(M1)***

$((z - 3) - \mathrm{i})((z - 3) + \mathrm{i})(z - 2)(z - 4)$ ***(A1)***

$\left(z^2 - 6z + 10\right)\left(z^2 - 6z + 8\right) = z^4 - 12z^3 + \ldots$

$p = -12$ ***A1***

[3 marks]

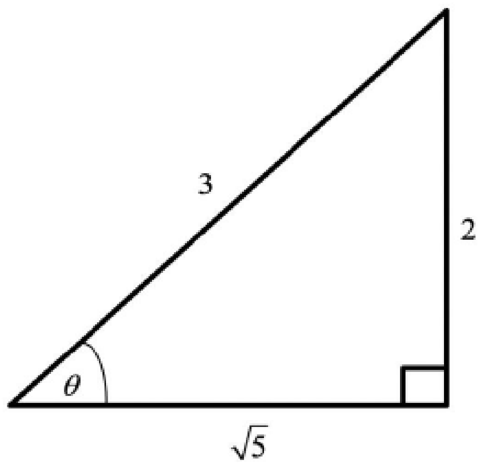
11. [Maximum mark: 4] 21M.1.AHL.TZ1.6

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$. [4]

Markscheme

METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 2

Attempt to use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ ***M1***

$1 + \cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

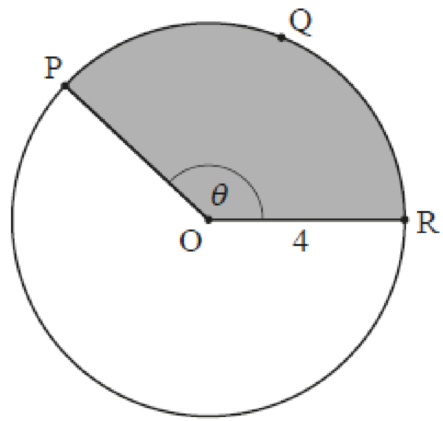
Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

12. [Maximum mark: 6] 23M.1.AHL.TZ2.1

The following diagram shows a circle with centre O and radius $4\ \mathrm{cm}$.

diagram not to scale



The points P, Q and R lie on the circumference of the circle and $\mathrm{P\hat{O}R} = \theta$, where θ is measured in radians.

The length of arc PQR is $10\ \mathrm{cm}$.

(a) Find the perimeter of the shaded sector. [2]

Markscheme

attempts to find perimeter *(M1)*

$\mathrm{arc} + 2 \times \mathrm{radius}$ OR $10 + 4 + 4$

$= 18\,(\mathrm{cm})$ *A1*

[2 marks]

Examiners report

SL:

Nearly all candidates performed well on this question, with most earning full marks. A few candidates tried to work in degrees, which often led to problems finding the answers in parts (b) and (c).

HL:

This question was carried out correctly by almost all students. One or two weaker students attempted to work in degrees, leading to some confusion and a considerable loss of marks in some cases.

(b) Find θ. [2]

Markscheme

$10 = 4\theta$ ***(A1)***

$\theta = \frac{10}{4}\left(= \frac{5}{2}, 2.5\right)$ ***A1***

[2 marks]

(c) Find the area of the shaded sector. [2]

Markscheme

area $= \frac{1}{2}\left(\frac{10}{4}\right)\left(4^2\right)\ \left(= 1.25 \times 16\right)$ ***(A1)***

$= 20\left(\text{cm}^2\right)$ ***A1***

[2 marks]

13. [Maximum mark: 5] 22N.1.AHL.TZ0.8

Let $f(x) = \cos(x - k)$, where $0 \leq x \leq a$ and $a,\ k \in \mathbb{R}^+$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists.

[2]

Markscheme

$a = \frac{\pi}{2}$ ***A2***

Note: For sinusoidal graph through the origin seen with incorrect a, or use of horizontal line test with incorrect a, award ***A1A0***

[2 marks]

Examiners report

While the first two parts of this question proved very accessible, incorrect answers were usually seen in part c). Some candidates made a reasonable start in attempting to sketch one or two sinusoidal functions but struggled to make further progress. Seemingly only the best candidates were able to gain full marks from this question part.

(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

Markscheme

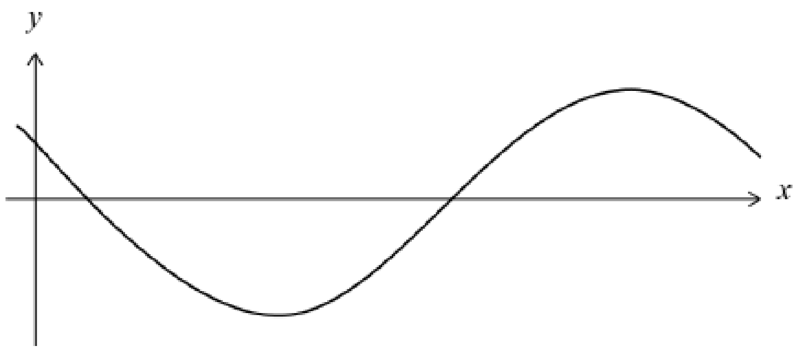
$a = \pi$ ***A1***

[1 mark]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k.

[2]

Markscheme



sketch showing sinusoidal shape decreasing as it crosses the y-axis

(below or above the origin) ***(A1)***

$a = k - \pi$ ***A1***

[2 marks]

14. [Maximum mark: 6] 22M.1.AHL.TZ1.9

Consider the complex numbers $z_1 = 1 + b\text{i}$ and $z_2 = \left(1 - b^2\right) - 2b\text{i}$, where $b \in \mathbb{R},\ b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

Markscheme

$z_1 z_2 = (1 + b\text{i})\left(\left(1 - b^2\right) - (2b)\text{i}\right)$

$= \left(1 - b^2 - 2\text{i}^2 b^2\right) + \text{i}\left(-2b + b - b^3\right)$ ***M1***

$= \left(1 + b^2\right) + \text{i}\left(-b - b^3\right)$ ***A1A1***

Note: Award ***A1*** for $1 + b^2$ and A1 for $-b\text{i} - b^3\text{i}$.

[3 marks]

Examiners report

Part (a) was generally well done with many completely correct answers seen. Part (b) proved to be challenging with many candidates incorrectly equating the ratio of their imaginary and real parts to $\frac{\pi}{4}$ instead of $\tan\frac{\pi}{4}$. Stronger candidates realized that when $\theta=\frac{\pi}{4}$, it forms an isosceles right-angled triangle and equated the real and imaginary parts to obtain the value of b.

(b) Hence, given that $\arg(z_1z_2)=\frac{\pi}{4}$, find the value of b. [3]

Markscheme

$\arg(z_1z_2)=\arctan\left(\frac{-b-b^3}{1+b^2}\right)=\frac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b)=\frac{\pi}{4}$ (since $1+b^2\neq 0$, for $b\in\mathbb{R}$) ***A1***

OR

$-b-b^3=1+b^2$ (or equivalent) ***A1***

THEN

$b=-1$ ***A1***

[3 marks]

15. [Maximum mark: 6] 25M.1.AHL.TZ1.6

The line L_1 has vector equation $\boldsymbol{r} = 4\boldsymbol{i} - \boldsymbol{k} + \lambda(a\boldsymbol{j} + \boldsymbol{k})$, where a, $\lambda \in \mathbb{R}$.

The line L_2 has vector equation $\boldsymbol{r} = \boldsymbol{i} - b\boldsymbol{k} + \mu(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$, where b, $\mu \in \mathbb{R}$.

The lines L_1 and L_2 are perpendicular and intersect at a unique point.

Find the value of a and the value of b. [6]

Markscheme

direction vectors are $a\boldsymbol{j} + \boldsymbol{k}\left(= \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}\right)$ and $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}\left(= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right)$

(A1)

recognition that the scalar product of the direction vectors is 0 ***(M1)***

$$\begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (= 2a + 3) = 0$$

$a = -\frac{3}{2}$ ***A1***

at point of intersection $4 = 1 + \mu$, $-\frac{3}{2}\lambda = 2\mu$ and $-1 + \lambda = -b + 3\mu$ ***(A1)***

attempt to solve 3 equations in μ, λ and b, derived from the point of intersection, to find μ, λ and b. ***(M1)***

$\mu = 3, \lambda = -4$

$b = 14$ ***A1***

[6 marks]

Examiners report

Those students who correctly wrote the given equations in column vector form often produced fully correct solutions, readily recognising the direction vectors and that the scalar product of these was zero. Some students seemed confused by the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ notation in the question.

16. [Maximum mark: 4] 20N.1.AHL.TZ0.H_1

A discrete random variable X has the probability distribution given by the following table.

x	0	1	2	3
$\mathrm{P}(X=x)$	p	$\frac{1}{4}$	$\frac{1}{6}$	q

Given that $\mathrm{E}(X) = \frac{19}{12}$, determine the value of p and the value of q. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\mathrm{E}(X) = (0 \times p) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{6}\right) + 3q\left(= \frac{19}{12}\right)$ ***(M1)***

$\left(\Rightarrow \frac{1}{4} + \frac{1}{3} + 3q = \frac{19}{12}\right)$

$q = \frac{1}{3}$ ***A1***

$p + \frac{1}{4} + \frac{1}{6} + q = 1$ ***(M1)***

$\left(\Rightarrow p + q = \frac{7}{12}\right)$

$p = \frac{1}{4}$ ***A1***

[4 marks]

17. [Maximum mark: 6] 25N.1.AHL.TZ3.2

Events A and B are such that $\mathrm{P}(A) = \frac{1}{2}$, $\mathrm{P}(B) = \frac{1}{3}$ and $\mathrm{P}(A|B) = \frac{1}{4}$.

(a) Find the probability that both events A and B occur. [2]

Markscheme

attempt to use $\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\mathrm{P}(B)\left(= \frac{1}{4} \times \frac{1}{3}\right)$ ***(M1)***

$= \frac{1}{12}$ ***A1***

[2 marks]

Examiners report

SL:

In part (a) of this question, many candidates did not recognize that finding the probability that both events A and B occur would require them to find the intersection, and many attempted to find the union instead. Of the candidates who recognized the need to find the intersection, many incorrectly assumed the events were independent, and simply multiplied $\mathrm{P}(A)$ and $\mathrm{P}(B)$, therefore obtaining an incorrect value. Few candidates earned full marks in part (b) of this question, which asked candidates to find $\mathrm{P}(A|B')$. Many candidates earned a mark for finding $\mathrm{P}(B') = \frac{2}{3}$, but very few were able to correctly find the value of $\mathrm{P}(A \cap B')$ necessary for the final calculation, with some candidates again incorrectly assuming independence and multiplying $\mathrm{P}(A)$ and $\mathrm{P}(B)$, and many others substituting their value of $\mathrm{P}(A \cap B)$ rather than attempting to find $\mathrm{P}(A \cap B')$, as needed.

HL:

Possibly due to a rushed start at the paper, attempts at part a seemed equally split between those giving an incorrect answer of $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ and the correct answer of $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$. Part b) was successfully answered by the better candidates, though a majority gained two of the four marks available through their use of $\mathrm{P}(B') = \frac{2}{3}$ and an attempt at conditional probability, often leading to a final incorrect answer of $\frac{1}{2}$. Though not a specific requirement, Venn diagrams were only occasionally employed.

(b) Find $\mathrm{P}(A|B\prime)$. [4]

Markscheme

$\mathrm{P}(B\prime) = \frac{2}{3}$ or $\mathrm{P}(A \cap B\prime) = \frac{5}{12}$ ***(A1)***

attempt to use $\mathrm{P}(A|B\prime) = \frac{\mathrm{P}(A \cap B\prime)}{\mathrm{P}(B\prime)}$ ***(M1)***

$= \frac{\left(\frac{5}{12}\right)}{\left(\frac{2}{3}\right)}$ ***(A1)***

$= \frac{5}{8}\left(= \frac{15}{24}\right)$ ***A1***

[4 marks]

18. [Maximum mark: 6] 24N.1.AHL.TZ0.2

Two events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.45$ and $\mathrm{P}(A \cup B) = 0.85$.

(a) Find $\mathrm{P}(A \cap B)$. [3]

Markscheme

use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

$\mathrm{P}(A \cap B) = 0.65 + 0.45 - 0.85$ (or equivalent) ***(A1)***

$= 0.25$ ***A1***

[3 marks]

Examiners report

Part (a) was usually answered correctly. Better students were able to answer part (b) with few problems, though this part did prove to be more discriminating. The correct formula for $P(A\prime|\ B\prime)$) was usually written down, but with $P(A\prime \cap B\prime)$ calculated incorrectly, with many suggesting $P(A\prime \cap B\prime) = 0.75$ leading to $P(A\prime|\ B\prime) = \frac{0.75}{0.55}$.

(b) Find $\mathrm{P}\left(A\prime \mid B\prime\right)$. [3]

Markscheme

$\mathrm{P}\left(A\prime \cap B\prime\right) = 0.15$ (may be seen in Venn diagram) ***(A1)***

attempt to substitute their values into $\mathrm{P}\left(A\prime \mid B\prime\right) = \frac{\mathrm{P}\left(A\prime \cap B\prime\right)}{\mathrm{P}\left(B\prime\right)}$ ***(M1)***

$\mathrm{P}\left(A\prime \mid B\prime\right) = \frac{0.15}{0.55}$

$= \frac{15}{55}\quad \left(= \frac{3}{11}\right)$ ***A1***

[3 marks]

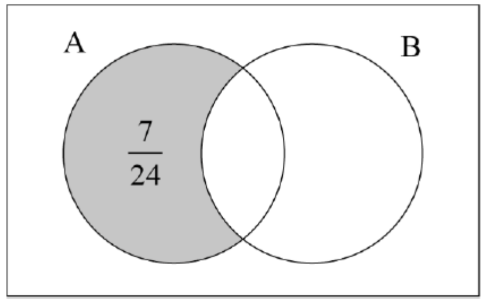
19. [Maximum mark: 7] 25M.1.AHL.TZ2.4

Events A and B are such that $\mathrm{P}(A \cup B) = \frac{5}{8}$ and $P(A \cap B\prime) = \frac{7}{24}$.

(a) Find $\mathrm{P}\left(B\right)$. [3]

Markscheme

attempt to use a Venn diagram **OR** law of addition ***(M1)***



$\mathrm{P}(A \cup B) = \mathrm{P}(A \cap B') + \mathrm{P}(B)$

$\frac{5}{8} = \frac{7}{24} + \mathrm{P}(B)$ ***(A1)***

$\mathrm{P}(B) = \frac{8}{24} \left(= \frac{1}{3}\right)$ ***A1***

[3 marks]

Examiners report

Part (a) was often answered successfully when candidates used Venn diagrams. In part (b), many identified the correct conditional probability expression but failed to recognise the need to solve for $\mathrm{P}(A)$, resulting in most being awarded only one mark.

(b) Given that events A and B are independent, find $\mathrm{P}(A' | B)$. [4]

Markscheme

METHOD 1: finding $\mathrm{P}(A)$

attempt to find $\mathrm{P}(A)$ ***M1***

$\mathrm{P}(A \cap B') = \mathrm{P}(A) \times \mathrm{P}(B')$ **OR**
$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \times \mathrm{P}(B)$

$\mathrm{P}(A) \times \frac{2}{3} = \frac{7}{24}$ **OR** $\frac{5}{8} = \frac{2}{3}\mathrm{P}(A) + \frac{1}{3}$

$\mathrm{P}(A) = \frac{7}{16}$ ***(A1)***

EITHER

$\mathrm{P}(A\prime\,|B) = \frac{\mathrm{P}(A\prime\cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A\prime)\times\mathrm{P}(B)}{\mathrm{P}(B)} = \mathrm{P}(A\prime)$ (seen anywhere) **OR** $1 - \frac{7}{16}$ ***(A1)***

OR

$\mathrm{P}(A\prime\,|B)\left(= \frac{\mathrm{P}(A\prime\cap B)}{\mathrm{P}(B)}\right) = \frac{\frac{9}{48}}{\frac{16}{48}}\left(= \frac{\frac{3}{16}}{\frac{1}{3}}\right)$ ***(A1)***

THEN

$\mathrm{P}(A\prime\,|B) = \frac{9}{16}$ ***A1***

METHOD 2: attempt to find $\mathrm{P}(A\cap B)$

attempt to use $\mathrm{P}(A\cap B) = \mathrm{P}(A)\times\mathrm{P}(B)$ ***M1***

$x = \left(x + \frac{7}{24}\right)\times\frac{1}{3}$

$x = \frac{7}{48}$ ***A1***

$\mathrm{P}(A\prime\,|B) = \mathrm{P}(B) - \mathrm{P}(A\cap B) + 1 - \mathrm{P}(A\cup B)$ **OR** $= \frac{1}{3} - \frac{7}{48} + 1 - \frac{5}{8}$ ***(A1)***

$\mathrm{P}(A\prime\,|B) = \frac{27}{48}\left(= \frac{9}{16}\right)$ ***A1***

[4 marks]

20. [Maximum mark: 8] 22M.1.AHL.TZ1.7

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{k}{\sqrt{4-3x^2}}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. [4]

Markscheme

attempt to integrate $\frac{k}{\sqrt{4-3x^2}}$ ***(M1)***

$= k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$ ***A1***

Note: Award ***(M1)A0*** for $\arcsin\left(\frac{\sqrt{3}}{2}x\right)$.
Condone absence of k up to this stage.

equating their integrand to 1 ***M1***

$k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]_0^1 = 1$

$k = \frac{3\sqrt{3}}{\pi}$ ***A1***

[4 marks]

Examiners report

Most candidates who attempted part (a) knew that the integrand must be equated to 1 and only a small proportion of these managed to recognize the standard integral involved here. The effect of 3 in $3x^2$ was missed by many resulting in very few completely correct answers for this part. Part (b) proved to be challenging for vast majority of the candidates and was poorly done in general. Stronger candidates who made good progress in part (a) were often successful in part (b) as well. Most candidates used a substitution, however many struggled to make progress using this approach. Often when using a substitution, the limits were unchanged. If the function was re-written in terms of *x*, this did not result in an error in the final answer.

(b) Find $\mathrm{E}(X)$. [4]

Markscheme

$\mathrm{E}\left(X\right) = \frac{3\sqrt{3}}{\pi} \int_0^1 \frac{x}{\sqrt{4-3x^2}} \mathrm{~d}\,x$ ***A1***

Note: Condone absence of limits if seen at a later stage.

EITHER

attempt to integrate by inspection ***(M1)***

$= \frac{3\sqrt{3}}{\pi} \times -\frac{1}{6} \int -6x\left(4 - 3x^2\right)^{-\frac{1}{2}} \mathrm{~d}\,x$

$= \frac{3\sqrt{3}}{\pi}\left[-\frac{1}{3}\sqrt{4 - 3x^2}\right]_0^1$ ***A1***

Note: Condone the use of k up to this stage.

OR

for example, $u = 4 - 3x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -6x$

Note: Other substitutions may be used. For example $u = -3x^2$.

$= -\frac{\sqrt{3}}{2\pi} \int_4^1 u^{-\frac{1}{2}} \mathrm{~d}\,u$ ***M1***

Note: Condone absence of limits up to this stage.

$= -\frac{\sqrt{3}}{2\pi}\left[2\sqrt{u}\right]_4^1$ ***A1***

Note: Condone the use of k up to this stage.

THEN

$= \frac{\sqrt{3}}{\pi}$ ***A1***

Note: Award ***A0M1A1A0*** for their $k\left[-\frac{1}{3}\sqrt{4-3x^2}\right]$ or $k\left[-2\sqrt{u}\right]$ for working with incorrect or no limits.

[4 marks]

21. [Maximum mark: 4] 25M.1.AHL.TZ1.3

Find the area completely enclosed by the curves $y = \mathrm{e}^x$, $y = -\mathrm{e}^x$, and the lines $x = -1$ and $x = 1$. [4]

Markscheme

METHOD 1

attempt to set up integral $\mathrm{e}^x - (-\mathrm{e}^x) = 2\mathrm{e}^x$ or $2\mathrm{e}^x$ and then double ***(M1)***

$\int(\mathrm{e}^x - (-\mathrm{e}^x))\,\mathrm{d}\,x$ **OR** $2\int \mathrm{e}^x\,\mathrm{d}\,x$

$= 2\int_{-1}^{1} \mathrm{e}^x\,\mathrm{d}\,x$

$= 2[\mathrm{e}^x]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into their integrated function and subtract ***(M1)***

$= 2\left(\mathrm{e} - \frac{1}{\mathrm{e}}\right)$, $2\mathrm{e} - \frac{2}{\mathrm{e}}$, $2\mathrm{e} - 2\mathrm{e}^{-1}$ ***A1***

METHOD 2

$\int_{-1}^{1} \mathrm{e}^x\,\mathrm{d}\,x = [\mathrm{e}^x]_{-1}^{1}$ and $\int_{-1}^{1} -\mathrm{e}^x\,\mathrm{d}\,x = [-\mathrm{e}^x]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into both their integrated functions and subtract ***(M1)***

$e^1 - e^{-1}$ and $-e^1 - \left(-e^{-1}\right)$

subtracts their two integrals in correct order ***(M1)***

$e^1 - e^{-1} - \left(-e^1 + e^{-1}\right)$

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

[4 marks]

Examiners report

SL:

Mixed response. Most students understood they needed to find the two areas using integration and substituted the correct limits. Sign errors were sometimes seen. Drawing a sketch of the two functions often supported understanding of the question.

HL:

There were many fully correct answers especially for those students who found the difference between the curves initially. Those students who split the area up into two integrals often introduced sign errors and there were some incorrect graphs which resulted in using incorrect limits.

22. [Maximum mark: 5] EXN.1.AHL.TZ0.6

Use l'Hôpital's rule to determine the value of $\lim_{x \to 0}\left(\frac{2x \cos(x^2)}{5 \tan x}\right)$. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA

course. There may be minor differences in formatting compared to formal exam papers.

attempts to apply l'Hôpital's rule on $\lim_{x\to 0}\left(\frac{2x\cos(x^2)}{5\tan x}\right)$ **M1**

$=\lim_{x\to 0}\left(\frac{2\cos(x^2)-4x^2\sin(x^2)}{5\sec^2 x}\right)$ **M1A1A1**

Note: Award **M1** for attempting to use product and chain rule differentiation on the numerator, **A1** for a correct numerator and **A1** for a correct denominator. The awarding of **A1** for the denominator is independent of the **M1**.

$=\frac{2}{5}$ **A1**

[5 marks]

23. [Maximum mark: 5] 25M.1.AHL.TZ1.1

Consider the function $f(x)=\frac{4x^3}{3}-16x$, where $x\in\mathbb{R}$.

The graph of $y=f(x)$ has a local minimum point at $(p,\ q)$ where $p>0$.

Find the value of p and the value of q. [5]

Markscheme

$f\prime(x)=4x^2-16$ ***A1***

sets their derivative equal to zero ***(M1)***

$4x^2-16=0, (x=\pm 2)$

$p=2$ (accept $x=2$) ***A1***

substitutes their **positive** p into $f(x)$ ***(M1)***

$y = \frac{4(2^3)}{3} - 16(2) \left(= \frac{32}{3} - 32 = -\frac{64}{3}\right)$

$q = -\frac{64}{3}$ (accept $y = \frac{-64}{3}$) ***A1***

[5 marks]

Examiners report

SL:

Many fully correct solutions seen. Most students understood they needed to differentiate and set the derivative equal to zero to find p. A number of students used the quotient rule to differentiate $\frac{4x^3}{3}$, which was inefficient use of their time. A few students failed to notice that the positive value of p should be used as stated in the question. Most realized that they should substitute p into the original equation but struggled with the fraction work.

HL:

Most students found this question straightforward, although there were a few calculation errors. Some students spent time verifying that they had found the minimum point by taking the second differential, which was not required.

24. [Maximum mark: 7] 24M.1.AHL.TZ1.8

(a) Find the first two non-zero terms in the Maclaurin series of

(a.i) $\sin\left(x^2\right)$; [2]

Markscheme

$\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \ldots \left(= x^2 - \frac{x^6}{6} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[2 marks]

Examiners report

In part (a) (i), many students correctly used the Maclaurin series for $\sin\, x$ given in the formula booklet to obtain the series for $\sin\, \left(x^2\right)$ as required. In part (a)(ii), a surprisingly large number of students chose to square each term individually rather than the entire series. Many students did not attempt part (b) and the response was generally poor for this part with very few correct answers seen.

(a.ii) $\sin^2\, \left(x^2\right)$. [3]

Markscheme

METHOD 1

attempt to square their series for $\sin\, \left(x^2\right)$ ***(M1)***

$\left(\sin\, \left(x^2\right)\right)^2 = \left(x^2 - \frac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\frac{x^6}{3!}\right)^2 + \ldots$.

$= x^4 - \frac{2x^8}{3!} + \ldots \left(x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\, \left(x^2\right) = \frac{1-\cos\, \left(2x^2\right)}{2}$ ***(M1)***

$$\sin^2\left(x^2\right) = \frac{1}{2}\left(1 - \left(1 - \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^4}{4!}\right)\right)$$

$$= x^4 - \frac{8x^8}{4!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$$ ***A1A1***

Note: Award ***A1*** for each term.

[3 marks]

(b) Hence, or otherwise, find the first two non-zero terms in the Maclaurin series of $4x\,\sin\left(x^2\right)\,\cos\left(x^2\right)$. [2]

Markscheme

METHOD 1

recognition that $4x\,\sin\left(x^2\right)\,\cos\left(x^2\right) = \frac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$$= 4x^3 - \frac{8x^7}{3} + \ldots$$ ***A1***

METHOD 2

recognition that $4x\,\sin\left(x^2\right)\,\cos\left(x^2\right) = 2x\,\sin\left(2x^2\right)$ ***(M1)***

$$= 2x\left(2x^2 - \frac{\left(2x^2\right)^3}{3!} + \ldots\right)$$

$$= 4x^3 - \frac{8x^7}{3} + \ldots$$ ***A1***

METHOD 3

$$4x\,\sin\left(x^2\right)\,\cos\left(x^2\right)$$

$$= 4x\left(x^2 - \frac{x^6}{3!} + \ldots\right)\left(1 - \frac{x^4}{2!} + \ldots\right)$$ ***(A1)***

$= 4x^3 - \frac{8x^7}{3} + \ldots$ ***A1***

METHOD 4

recognition that $2x\cos\left(x^2\right) = \frac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$= 2\left(x^2 - \frac{x^6}{3!} + \ldots\right)\left(2x - \frac{6x^5}{2!} + \ldots\right)$

$= 4x^3 - \frac{8x^7}{3} + \ldots$ ***A1***

[2 marks]

25. [Maximum mark: 8] 25M.1.AHL.TZ1.9

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y}$, where $x > 0$ and $y \neq -x$.

It is given that $y = 0$ when $x = 2$.

By using the substitution $y = vx$, show that the solution of the differential equation is $x^2 - 2xy - y^2 = 4$. [8]

Markscheme

$y = vx$, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{x-vx}{x+vx}\left(= \frac{1-v}{1+v}\right)$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{1-v}{1+v}$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-v}{1+v} - v\left(= \frac{1-2v-v^2}{1+v}\right)$

attempt to separate variables and form two integrals ***(M1)***

$\int \frac{1+v}{1-2v-v^2}\ \mathrm{d}\ v = \int \frac{1}{x}\ \mathrm{d}\ x$

$\int \frac{1}{x}\ \mathrm{d}\ x = \ln x + c$ ***A1***

use of substitution or inspection to integrate $\frac{1+v}{1-2v-v^2}$ or equivalent ***(M1)***

$u = 1 - 2v - v^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}v} = -2 - 2v = -2(1+v)$

$\int \frac{1+v}{1-2v-v^2}\ \mathrm{d}\ v = -\frac{1}{2}\ln\left|1-2v-v^2\right|$ **OR**
$-\frac{1}{2}\ln\left|v^2+2v-1\right| = \ln|x| + c$ ***A1***

$-\frac{1}{2}\ln\left|1-2\frac{y}{x}-\frac{y^2}{x^2}\right| - \ln|x| = c$ **OR**
$-\frac{1}{2}\ln\left|\frac{y^2}{x^2}+2\frac{y}{x}-1\right| - \ln|x| = c$

EITHER

attempt to substitute $x = 2$ and either $y = 0$ or $v = 0$ to find a constant c ***(M1)***

$c = -\ln 2$

$-\frac{1}{2}\ln\left|1-2\frac{y}{x}-\frac{y^2}{x^2}\right| - \ln|x| = -\ln 2$ **OR**
$-\frac{1}{2}\ln\left|\frac{y^2}{x^2}+2\frac{y}{x}-1\right| - \ln|x| = -\ln 2$ ***A1***

OR

$x^2\left|1-2\frac{y}{x}-\frac{y^2}{x^2}\right| = A$ **OR** $x^2\left|\frac{y^2}{x^2}+2\frac{y}{x}-1\right| = A$ ***A1***

attempt to substitute $x = 2$ either $y = 0$ or $v = 0$ to find a constant A ***(M1)***

THEN

$x^2\left|1-2\frac{y}{x}-\frac{y^2}{x^2}\right| = 4$ **OR** $x^2\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right) = 4$ ***A1*** checking boundary values confirms $x^2 - 2xy - y^2 = 4$ ***AG***

Note: Condone absence of absolute value signs even if removed incorrectly until the final ***A1*** mark where they must be seen or have been removed to form a correct equation.

[8 marks]

Examiners report

For those students who were familiar with the process most applied the substitution given although there were some algebraic errors. Some managed to separate the variables correctly but again there were algebraic errors. Those that did separate the variables correctly generally integrated effectively using recognition or substitution. A small number of students, by removing the modulus incorrectly, introduced a sign error.

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