

Paper 1 May 2024 [110 marks]

1. [Maximum mark: 6]

24M.1.AHL.TZ1.1

Claire rolls a six-sided die 16 times.

The scores obtained are shown in the following frequency table.

Score	Frequency
1	p
2	q
3	4
4	2
5	0
6	3

It is given that the mean score is 3.

(a) Find the value of p and the value of q .

[5]

Markscheme

attempt to form equation for the sum of frequencies = 16 or mean = 3 (M1)

$$p + q + 4 + 2 + 3 = 16 (\Rightarrow p + q = 7) \quad A1$$

$$\frac{p+2q+12+8+18}{16} = 3 (\Rightarrow p + 2q = 10) \quad \text{OR} \quad \frac{p+2q+12+8+18}{9+p+q} = 3 (\Rightarrow 2p + q = 11) \quad A1$$

attempt to eliminate one variable from their equations (M1)

$$p + 2(7 - p) + 38 = 48 \quad \text{OR} \quad 2(7 - q) + q = 11$$

$$p = 4 \quad \text{and} \quad q = 3 \quad A1$$

Note: Award M1A0A0M0A1 for $p = 4$, $q = 3$ with no working.

[5 marks]

Each of Claire's scores is multiplied by 10 in order to determine the final score for a game she is playing.

(b) Write down the mean final score.

[1]

Markscheme
mean final score = 30 <i>A1</i>
<i>[1 mark]</i>

2. [Maximum mark: 5]

24M.1.AHL.TZ1.2

It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10} \left(\frac{1}{a}\right)$;

[2]

Markscheme
$\log_{10} 1 - \log_{10} a$ OR $\log_{10} a^{-1} = -\log_{10} a$ OR $\log_{10} 10^{-\frac{1}{3}}$ OR $10^x = \frac{1}{10^{\frac{1}{3}}}$
<i>(A1)</i>
$= -\frac{1}{3}$ <i>A1</i>
<i>[2 marks]</i>

(b) $\log_{1000} a$.

[3]

Markscheme
$\frac{\log_{10} a}{\log_{10} 1000}$ OR $\frac{1}{3} \log_{1000} 10$ OR $\log_{1000} \sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}} = 1000^x (= (10^3)^x)$
<i>(A1)</i>
$\frac{\log_{10} a}{3}$ OR $\frac{1}{3} \log_{1000} 1000^{\frac{1}{3}}$ OR $\log_{1000} 1000^{\frac{1}{9}}$ OR $3x = \frac{1}{3}$ <i>(A1)</i>
$= \frac{1}{9}$ <i>A1</i>
<i>[3 marks]</i>

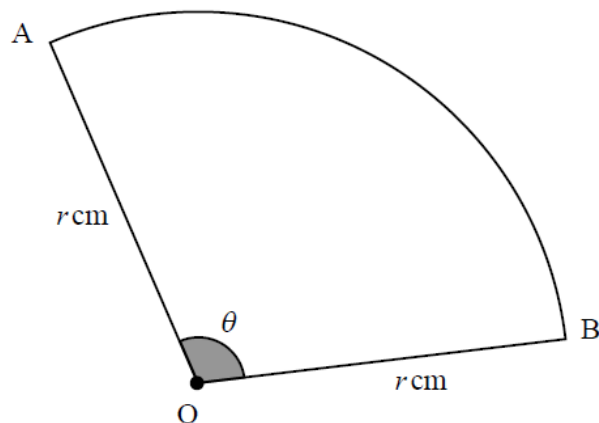
3. [Maximum mark: 8]

24M.1.AHL.TZ1.3

Points **A** and **B** lie on the circumference of a circle of radius r cm with centre at **O**.

The sector OAB is shown on the following diagram. The angle \widehat{AOB} is denoted as θ and is measured in radians.

diagram not to scale



The perimeter of the sector is 10 cm and the area of the sector is 6.25 cm^2 .

(a) Show that $4r^2 - 20r + 25 = 0$.

[4]

Markscheme

$$2r + r\theta = 10 \quad A1$$

$$\frac{1}{2}r^2\theta = 6.25 \quad A1$$

attempt to eliminate θ to obtain an equation in r *M1*

correct intermediate equation in r *A1*

$$10 - 2r = \frac{25}{2r} \quad \text{OR} \quad \frac{10}{r} - 2 = \frac{25}{2r^2} \quad \text{OR} \quad \frac{1}{2}r^2\left(\frac{10}{r} - 2\right) = 6.25 \quad \text{OR}$$

$$1.25 + 2r^2 = 10r$$

$$4r^2 - 20r + 25 = 0 \quad AG$$

[4 marks]

(b) Hence, or otherwise, find the value of r and the value of θ .

[4]

Markscheme

attempt to solve quadratic by factorizing or use of formula or completing the square *(M1)*

$$(2r - 5)^2 = 0 \quad \text{OR} \quad r = \frac{20 \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)} \quad \left(= \frac{20 \pm \sqrt{400 - 400}}{8} \right)$$

$$r = \frac{5}{2} \quad A1$$

attempt to substitute their value of r into their perimeter or area equation (M1)

$$\theta = \frac{10 - 2\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)} \text{ or } \theta = \frac{25}{2\left(\frac{5}{2}\right)^2}$$

$$\theta = 2 \quad A1$$

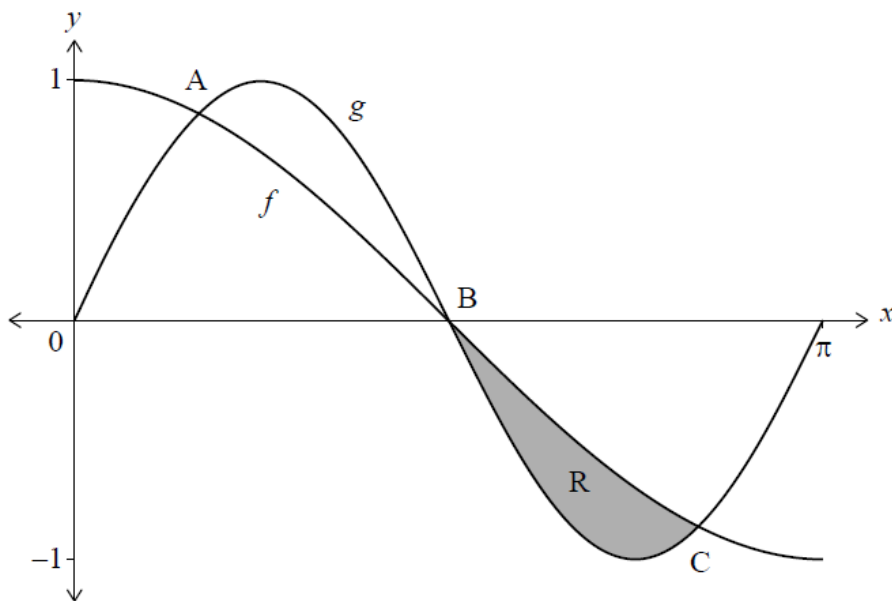
[4 marks]

4. [Maximum mark: 7]

24M.1.AHL.TZ1.4

Consider the functions $f(x) = \cos x$ and $g(x) = \sin 2x$, where $0 \leq x \leq \pi$.

The graph of f intersects the graph of g at the point A, the point B $\left(\frac{\pi}{2}, 0\right)$ and the point C as shown on the following diagram.



(a) Find the x -coordinate of point A and the x -coordinate of point C.

[3]

Markscheme

recognising $\cos x = 2 \sin x \cos x$ (M1)

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) (A1)

x -coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ A1

Note: Award *(M1)(A1)A0* for solutions of 30° and 150° .

[3 marks]

The shaded region **R** is enclosed by the graph of f and the graph of g between the points **B** and **C**.

(b) Find the area of **R**.

[4]

Markscheme

METHOD 1

attempt to integrate $\pm(\cos x - \sin 2x)$ *(M1)*

$$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - \sin 2x) \, dx \text{ OR } \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - 2 \sin x \cos x) \, dx$$

$$= \left[\sin x + \frac{1}{2} \cos 2x \right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \text{ OR } = \left[\sin x - \sin^2 x \right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \quad \mathbf{A1}$$

Note: Award *A1* for \pm correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract *M1*

$$= \left(\sin \left(\frac{5\pi}{6} \right) + \frac{1}{2} \cos \left(\frac{5\pi}{3} \right) \right) - \left(\sin \left(\frac{\pi}{2} \right) + \frac{1}{2} \cos (\pi) \right) \text{ OR}$$
$$\left(\sin \left(\frac{5\pi}{6} \right) - \sin^2 \left(\frac{5\pi}{6} \right) \right) - \left(\sin \left(\frac{\pi}{2} \right) - \sin^2 \left(\frac{\pi}{2} \right) \right)$$

$$= \left(\frac{1}{2} + \frac{1}{4} \right) - \left(1 - \frac{1}{2} \right) \text{ OR } = \left(\frac{1}{2} - \frac{1}{4} \right) - (1 - 1)$$

$$\text{area} = \frac{1}{4} \quad \mathbf{A1}$$

Note: Award all corresponding marks as appropriate for finding the area between **A** and **B**.

Accept work done in degrees.

METHOD 2

$$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \, dx = \left[\sin x \right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \quad \text{and} \quad \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \, dx = \left[-\frac{1}{2} \cos 2x \right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \quad \mathbf{A1}$$

Note: Award **A1** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) **M1**

$$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right) \quad \text{and} \quad -\frac{1}{2} \cos\left(\frac{5\pi}{3}\right) + \frac{1}{2} \cos(\pi)$$

attempt to subtract the two integrals in either order (seen anywhere) **(M1)**

$$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right) \right) - \left(-\frac{1}{2} \cos\left(\frac{5\pi}{3}\right) + \frac{1}{2} \cos(\pi) \right) \quad \text{OR}$$

$$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \, dx - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \, dx$$

$$= \left(\frac{1}{2} - 1 \right) - \left(\frac{1}{4} - \frac{1}{2} \right) \quad \left(= -\frac{1}{4} \right)$$

$$\text{area} = \frac{1}{4} \quad \mathbf{A1}$$

Note: Award all corresponding marks as appropriate for finding the area between **A** and **B**.

Accept work done in degrees.

[4 marks]

5. [Maximum mark: 5]

24M.1.AHL.TZ1.5

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n - 1}{b}$, where $a, b \in \mathbb{Z}^+$.

[1]

Markscheme

$$S_n = \frac{10^n - 1}{9} \quad \mathbf{A1}$$

$$(a = 10, b = 9)$$

[1 mark]

(b) Hence, show that $S_1 + S_2 + S_3 + \dots + S_n = \frac{10(10^n - 1) - 9n}{81}$.

[4]

Markscheme

METHOD 1

$$S_1 + S_2 + S_3 + \dots + S_n$$

$$= \frac{10-1}{9} + \frac{10^2-1}{9} + \dots + \frac{10^n-1}{9} \quad \text{A1}$$

$$= \frac{10-1+10^2-1+10^3-1+\dots+10^n-1}{9} \quad \text{OR} \quad \frac{9(10-1+10^2-1+10^3-1+\dots+10^n-1)}{81}$$

attempt to use geometric series formula on powers of 10, and collect -1 's together **M1**

$$10 + 10^2 + 10^3 + \dots + 10^n = \frac{10(10^n-1)}{10-1} \quad \text{and} \quad -1 - 1 - 1 \dots = -n \quad \text{A1}$$

$$= \frac{\frac{10(10^n-1)}{10-1} - n}{9} \quad \text{OR} \quad \frac{9\left(\frac{10(10^n-1)}{10-1}\right) - 9n}{81} \quad \text{A1}$$

Note: Award **A1** for any correct intermediate expression.

$$= \frac{10(10^n-1) - 9n}{81} \quad \text{AG}$$

METHOD 2

attempt to create sum using sigma notation with S_n **M1**

$$\sum_{i=1}^n \frac{10^i-1}{9} \quad \left(= \frac{1}{9} \left(\sum_{i=1}^n 10^i - \sum_{i=1}^n 1 \right) \right)$$

$$\sum_{i=1}^n 10^i = \frac{10(10^n-1)}{9} \quad \text{A1}$$

$$\sum_{i=1}^n 1 = n \quad \text{A1}$$

$$= \frac{1}{9} \left(\frac{10(10^n-1)}{9} - n \right) \quad \text{OR} \quad \frac{1}{9} \left(\frac{10(10^n-1) - 9n}{9} \right) \quad \text{A1}$$

$$= \frac{10(10^n-1) - 9n}{81} \quad \text{AG}$$

METHOD 3

let $P(n)$ be the proposition that $S_1 + S_2 + S_3 + \dots + S_n = \frac{10(10^n - 1) - 9n}{81}$

considering $P(1)$:

$$\text{LHS} = S_1 = \frac{10^1 - 1}{9} = 1 \quad \text{and} \quad \text{RHS} = \frac{10(10^1 - 1) - 9(1)}{81} = 1 \quad \text{and so } P(1) \text{ is true} \quad \mathbf{R1}$$

$$\text{assume } P(k) \text{ is true i.e. } S_1 + S_2 + S_3 + \dots + S_k = \frac{10(10^k - 1) - 9k}{81} \quad \mathbf{M1}$$

Note: Do not award **M1** for statements such as "let $n = k$ " or " $n = k$ is true". Subsequent marks after this **M1** are independent of this mark and can be awarded.

considering $P(k + 1)$:

$$\begin{aligned} S_1 + S_2 + S_3 + \dots + S_{k+1} &= \frac{10(10^k - 1) - 9k}{81} + \frac{10^{k+1} - 1}{9} \\ &= \frac{10^{k+1} - 10 - 9k + 9(10^k - 1) - 9}{81} \quad \mathbf{A1} \\ &= \frac{10(10^{k+1} - 1) - 9(k+1)}{81} \end{aligned}$$

$P(k + 1)$ is true whenever $P(k)$ is true and $P(1)$ is true, so $P(n)$ is true $\mathbf{R1}$

(for all integers $n \geq 1$)

Note: To obtain the final **R1**, the first **R1** and **A1** must have been awarded.

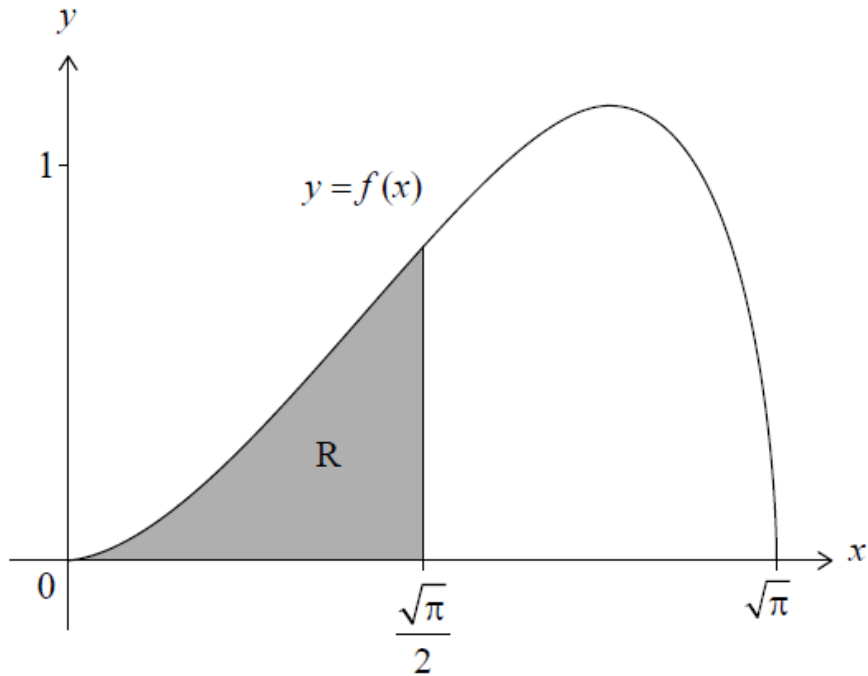
[4 marks]

6. [Maximum mark: 6]

24M.1.AHL.TZ1.6

The function f is defined as $f(x) = \sqrt{x \sin(x^2)}$, where $0 \leq x \leq \sqrt{\pi}$.

Consider the shaded region **R** enclosed by the graph of f , the x -axis and the line $x = \frac{\sqrt{\pi}}{2}$, as shown in the following diagram.



The shaded region R is rotated by 2π radians about the x -axis to form a solid.

[6]

Show that the volume of the solid is $\frac{\pi(2-\sqrt{2})}{4}$.

Markscheme

METHOD 1

attempt to find an integral involving π and the square of $f(x)$ **M1**

Note: Condone incorrect or absent limits for this **M1**.

$$\pi \int_0^{\frac{\sqrt{\pi}}{2}} (f(x))^2 dx$$

$$\pi \int_0^{\frac{\sqrt{x}}{2}} x \sin(x^2) dx \quad \mathbf{A1}$$

EITHER

attempt to use integration by substitution **M1**

$$\frac{\pi}{2} \int_0^{\frac{\pi}{4}} \sin(u) \, du$$

Note: Award *M1* for $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$$= \left[-\frac{\pi}{2} \cos(u) \right]_0^{\frac{\pi}{4}} \quad \mathbf{A1}$$

OR

attempt to integrate by inspection *(M1)*

$$\frac{\pi}{2} \int_0^{\frac{\sqrt{\pi}}{2}} 2x \sin(x^2) \, dx \quad \text{OR} \quad \frac{\pi}{2} \int_0^{\frac{\sqrt{\pi}}{2}} \sin(x^2) \, d(x^2)$$

$$= \left[-\frac{\pi}{2} \cos(x^2) \right]_0^{\frac{\sqrt{\pi}}{2}} \quad \mathbf{A1}$$

Note: Condone incorrect or absent limits for *M1*.

The correct limits may be seen or implied by later work for the *A1*.

THEN

$$= \left(-\frac{\pi}{2} \cos\left(\frac{\pi}{4}\right) \right) - \left(-\frac{\pi}{2} \cos(0) \right) \quad (\text{or equivalent}) \quad \mathbf{A1}$$

$$= -\frac{\pi}{2\sqrt{2}} + \frac{\pi}{2} \quad \text{OR} \quad -\frac{\pi\sqrt{2}}{4} + \frac{\pi}{2} \quad \text{OR} \quad \frac{\pi}{2} \left(-\frac{1}{\sqrt{2}} + 1 \right) \quad \text{OR} \quad \frac{\pi}{2} \left(-\frac{\sqrt{2}}{2} + 1 \right) \quad \mathbf{A1}$$

$$= \frac{\pi(2-\sqrt{2})}{4} \quad \mathbf{AG}$$

METHOD 2

attempt to find an integral involving π and the square of $f(x)$ *M1*

Note: Condone incorrect or absent limits for this *M1*.

$$\pi \int_0^{\frac{\sqrt{\pi}}{2}} (f(x))^2 dx$$

$$\pi \int_0^{\frac{\sqrt{\pi}}{2}} x \sin(x^2) dx \quad A1$$

attempt to use integration by substitution **M1**

$$u = \cos(x^2) \Rightarrow \frac{du}{dx} = -2x \sin(x^2)$$

Note: Award **M1** for $u = \cos(x^2)$

$$= -\frac{\pi}{2} \int_{-\frac{1}{\sqrt{2}}}^{-1} du$$

$$= \left[-\frac{\pi}{2} u \right]_{-\frac{1}{\sqrt{2}}}^{-1} \text{ (or equivalent)} \quad A1A1$$

Note: Condone incorrect or absent limits for **M1**.

A1 for $-\frac{\pi}{2}u$ and **A1** for both correct limits.

$$= \frac{\pi}{2} - \frac{\pi}{2\sqrt{2}} \quad \text{OR} \quad \frac{\pi}{2} - \frac{\pi\sqrt{2}}{4} \quad \text{OR} \quad \frac{\pi}{2} \left(1 - \frac{1}{\sqrt{2}} \right) \quad \text{OR} \quad \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) \quad A1$$

$$= \frac{\pi(2-\sqrt{2})}{4} \quad AG$$

[6 marks]

7. [Maximum mark: 7]

24M.1.AHL.TZ1.7

Using mathematical induction and the definition ${}^n C_r = \frac{n!}{r!(n-r)!}$, prove that

$$\sum_{r=1}^n r C_1 = {}^{n+1} C_2 \text{ for all } n \in \mathbb{Z}^+.$$

[7]

Markscheme

Base case $n = 1$: LHS = ${}^1C_1 = 1$ and RHS = ${}^2C_2 = 1$, so true for $n = 1$ **R1**

Note: Award **R0** if the value of 1C_1 and 2C_2 are not evaluated.

Subsequent marks can still be awarded.

assume true for $n = k$ ie $\sum_{r=1}^k {}^rC_1 = {}^{k+1}C_2$ for some $k \in \mathbb{Z}^+$ **M1**

Note: The assumption of truth must be clear.

Award **M0** for statements such as "let $n = k$ " or " $n = k$ is true".

Subsequent marks can still be awarded.

consider $n = k + 1$

$$\begin{aligned} \text{LHS} &= \sum_{r=1}^{k+1} {}^rC_1 \\ &= \sum_{r=1}^k {}^rC_1 + {}^{k+1}C_1 \quad (\text{M1}) \\ &= {}^{k+1}C_2 + {}^{k+1}C_1 \quad \text{OR} \quad \frac{(k+1)!}{2(k-1)!} + \frac{(k+1)!}{k!} \quad \text{A1} \end{aligned}$$

EITHER

attempt to cancel factorials and use a common denominator **M1**

$$= \frac{(k+1)k+2(k+1)}{2} \left(= \frac{(k+2)(k+1)}{2} \right)$$

OR

attempt to use a common denominator **M1**

$$= \frac{k(k+1)!}{2k!} + \frac{2(k+1)!}{2k!} \left(= \frac{(k+2)(k+1)!}{2k!} \right)$$

THEN

$$= \frac{(k+2)!}{2!k!} \left(= \frac{(k+2)!}{2!(k+2-2)!} \right) \quad \mathbf{A1}$$

$$= {}^{k+2}C_2$$

since true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$,

therefore true for all $n \in \mathbb{Z}^+$ **R1**

Note: Only award the final **R1** if 4 of the previous 6 marks have been awarded.

[7 marks]

Note: Throughout this question, condone presence of any additional terms once the first two correct terms are seen.

8. [Maximum mark: 7]

24M.1.AHL.TZ1.8

(a) Find the first two non-zero terms in the Maclaurin series of

(a.i) $\sin(x^2)$;

[2]

Markscheme

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \dots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \dots \left(= x^2 - \frac{x^6}{6} + \dots \right) \quad \mathbf{A1A1}$$

Note: Award **A1** for each term.

[2 marks]

(a.ii) $\sin^2(x^2)$.

[3]

Markscheme

METHOD 1

attempt to square their series for $\sin(x^2)$ **(M1)**

$$(\sin(x^2))^2 = \left(x^2 - \frac{x^6}{3!} + \dots\right)^2$$

Note: Award *M0* for $(x^2)^2 - \left(\frac{x^6}{3!}\right)^2 + \dots$

$$= x^4 - \frac{2x^8}{3!} + \dots \left(x^4 - \frac{x^8}{3} + \dots\right) \quad \mathbf{A1A1}$$

Note: Award *A1* for each term.

METHOD 2

attempt to use the identity $\sin^2(x^2) = \frac{1 - \cos(2x^2)}{2}$ *(M1)*

$$\sin^2(x^2) = \frac{1}{2} \left(1 - \left(1 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!}\right)\right)$$

$$= x^4 - \frac{8x^8}{4!} + \dots \left(= x^4 - \frac{x^8}{3} + \dots\right) \quad \mathbf{A1A1}$$

Note: Award *A1* for each term.

[3 marks]

- (b) Hence, or otherwise, find the first two non-zero terms in the Maclaurin series of $4x \sin(x^2) \cos(x^2)$.

[2]

Markscheme

METHOD 1

recognition that $4x \sin(x^2) \cos(x^2) = \frac{d((\sin(x^2))^2)}{dx}$ *(M1)*

$$= 4x^3 - \frac{8x^7}{3} + \dots \quad \mathbf{A1}$$

METHOD 2

recognition that $4x \sin(x^2) \cos(x^2) = 2x \sin(2x^2)$ (M1)

$$= 2x \left(2x^2 - \frac{(2x^2)^3}{3!} + \dots \right)$$

$$= 4x^3 - \frac{8x^7}{3} + \dots \quad A1$$

METHOD 3

$4x \sin(x^2) \cos(x^2)$

$$= 4x \left(x^2 - \frac{x^6}{3!} + \dots \right) \left(1 - \frac{x^4}{2!} + \dots \right) \quad (A1)$$

$$= 4x^3 - \frac{8x^7}{3} + \dots \quad A1$$

METHOD 4

recognition that $2x \cos(x^2) = \frac{d(\sin(x^2))}{dx}$ (M1)

$4x \sin(x^2) \cos(x^2)$

$$= 2 \left(x^2 - \frac{x^6}{3!} + \dots \right) \left(2x - \frac{6x^5}{2!} + \dots \right)$$

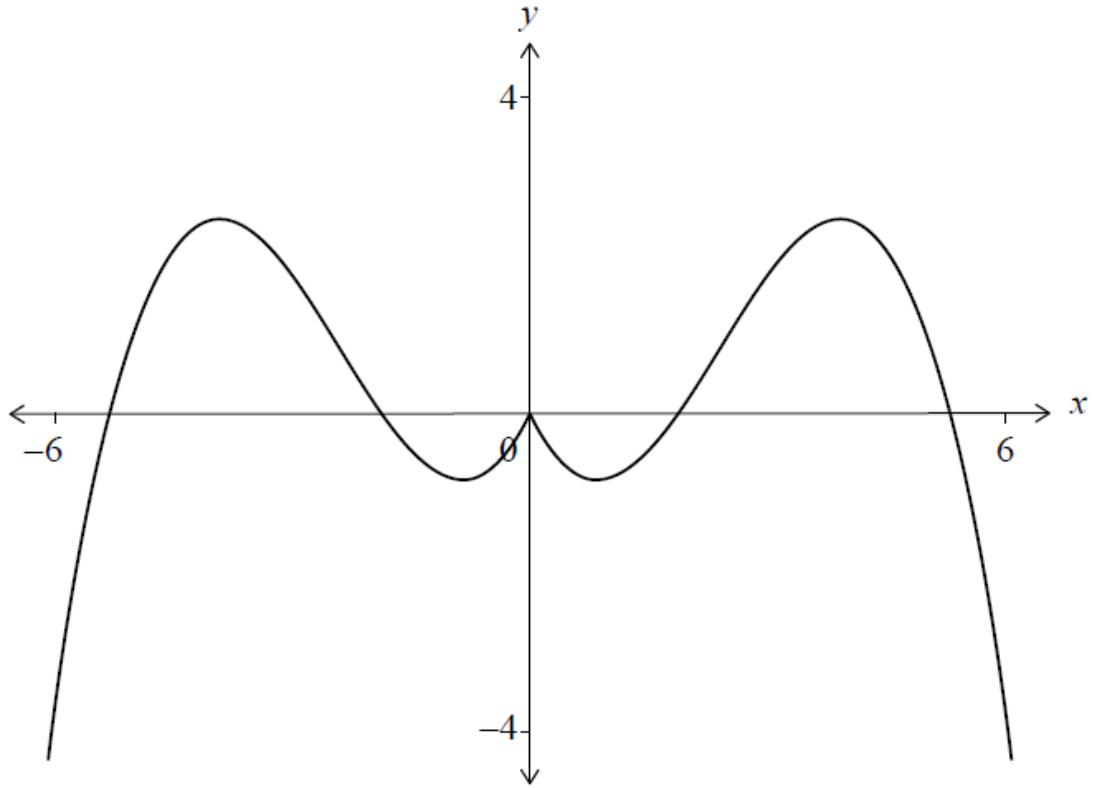
$$= 4x^3 - \frac{8x^7}{3} + \dots \quad A1$$

[2 marks]

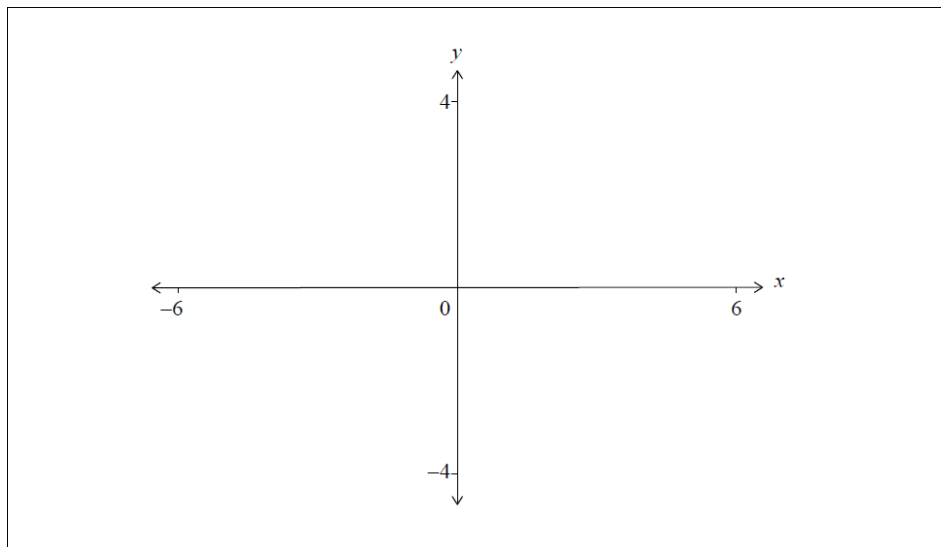
9. [Maximum mark: 6]

24M.1.AHL.TZ1.9

The graph of $y = f(|x|)$ for $-6 \leq x \leq 6$ is shown in the following diagram.



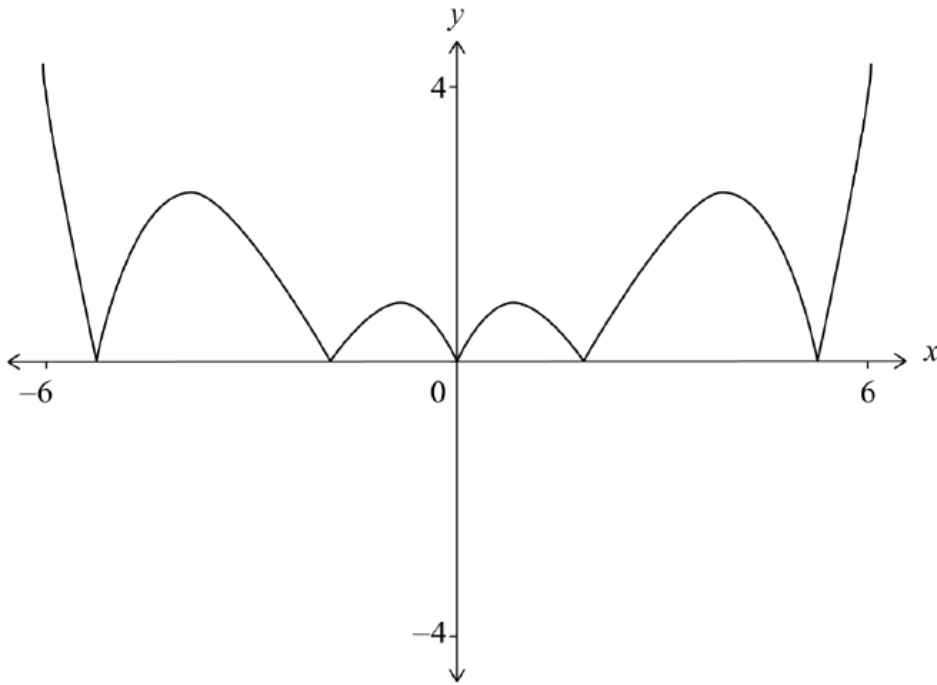
(a) On the following axes, sketch the graph of $y = |f(|x|)|$ for $-6 \leq x \leq 6$.



[2]

Markscheme

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reflection of all negative sections in x -axis (M1)

approximately correct graph with sharp points (cusps) at x -intercepts A1

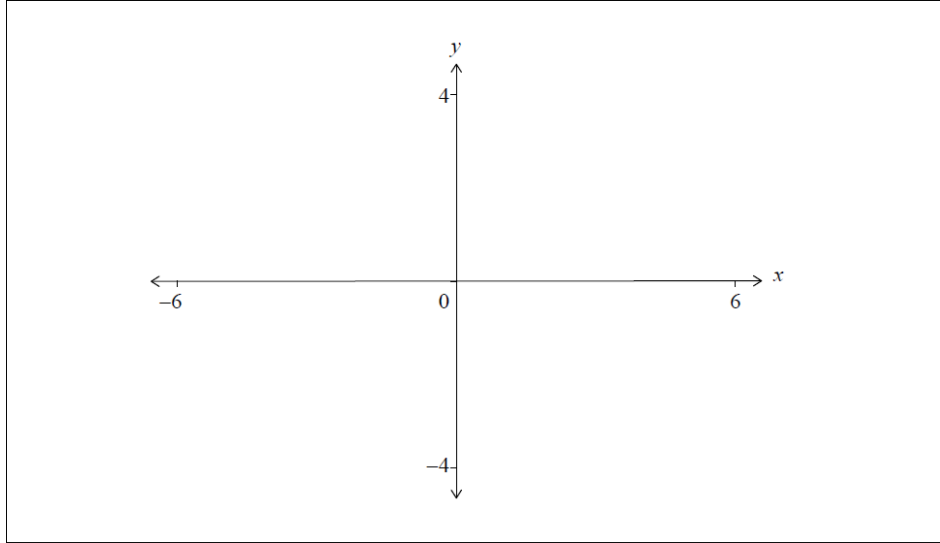
Note: Award A1 only if the heights of the maximum in the middle are lower than the heights of the maximum at the ends.

[2 marks]

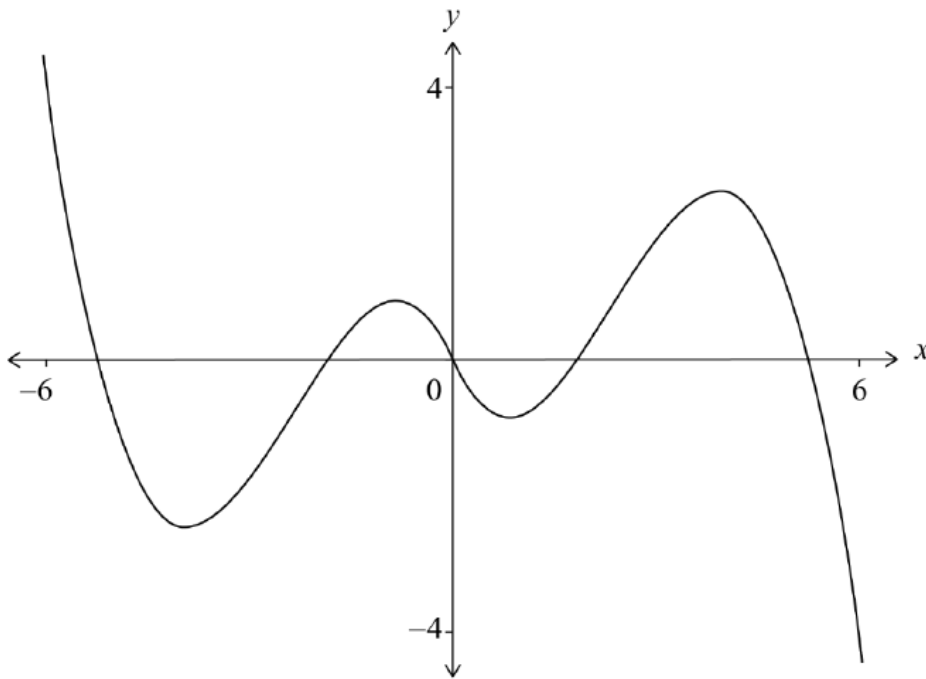
It is given that f is an odd function.

(b) On the following axes, sketch the graph of $y = f(x)$ for $-6 \leq x \leq 6$.

[2]



Markscheme

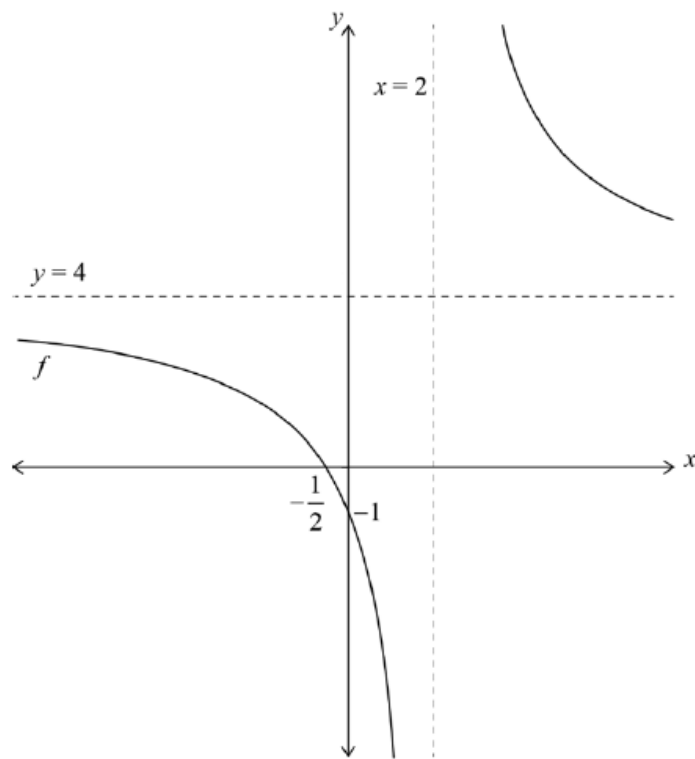


A1A1

Note: Award **A1** for right hand side unchanged and **A1** for rotation 180° about the origin.

[2 marks]

It is also given that $\int_0^4 f(|x|) dx = 1.6$.



vertical asymptote $x = 2$ sketched and labelled with correct equation **A1**

horizontal asymptote $y = 4$ sketched and labelled with correct equation **A1**

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x -axis, -1 on y -axis **A1A1**

two branches in correct opposite quadrants with correct asymptotic behaviour **A1**

Note: These marks may be awarded independently.

[5 marks]

(b) Write down the range of f .

[1]

Markscheme

$y \neq 4$ (or equivalent) **A1**

[1 mark]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p , where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$.

[1]

Markscheme

$$2 + \frac{5}{2} \text{ OR } \left(-\frac{1}{2}\right) + 2 \times \frac{5}{2} \text{ OR } \frac{-\frac{1}{2} + p}{2} = 2 \text{ OR } -4 = -p + \frac{1}{2} \quad \mathbf{A1}$$

$$p = \frac{9}{2} \quad \mathbf{AG}$$

[1 mark]

(d) Find the value of b and the value of c .

[3]

Markscheme

METHOD 1

attempt to substitute both roots to form a quadratic **(M1)**

EITHER

$$\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) \text{ OR } x^2 - \left(-\frac{1}{2} + \frac{9}{2}\right)x + \left(-\frac{1}{2} \times \frac{9}{2}\right)$$

$$= x^2 - 4x - \frac{9}{4} \quad \mathbf{A1A1}$$

$$\left(b = -4, c = -\frac{9}{4}\right)$$

Note: Award **A1** for each correct value. They may be embedded or stated explicitly.

OR

$$(2x + 1)(2x - 9) = 4\left(x^2 - 4x - \frac{9}{4}\right)$$

$$b = -4, c = -\frac{9}{4} \quad \mathbf{A1A1}$$

Note: Award **A1** for each correct value. They must be stated explicitly.

METHOD 2

$$-\frac{b}{2} = 2 \text{ OR } 4 + b = 0 \Rightarrow b = -4 \quad \mathbf{A1}$$

attempt to form a valid equation to find c using their b (M1)

$$\left(-\frac{1}{2}\right)^2 + -4\left(-\frac{1}{2}\right) + c = 0 \text{ OR } \left(\frac{9}{2}\right)^2 + -4\left(\frac{9}{2}\right) + c = 0$$

$$c = -\frac{9}{4} \quad \mathbf{A1}$$

METHOD 3

attempt to form two valid equations in b and c (M1)

$$\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + c = 0, \left(\frac{9}{2}\right)^2 + b\left(\frac{9}{2}\right) + c = 0$$

$$b = -4, c = -\frac{9}{4} \quad \mathbf{A1A1}$$

METHOD 4

attempt to write $g(x)$ in the form $(x - h)^2 + k$ and substitute for x , h and $g(x)$ (M1)

$$\left(-\frac{1}{2} - 2\right)^2 + k = 0 \Rightarrow k = -\frac{25}{4}$$

$$(x - 2)^2 - \frac{25}{4}$$

$$= x^2 - 4x - \frac{9}{4} \quad \mathbf{A1A1}$$

$$(b = -4, c = -\frac{9}{4})$$

Note: Award **A1** for each correct value. They may be embedded or stated explicitly.

[3 marks]

(e) Find the y -coordinate of the vertex of the graph of $y = g(x)$.

[2]

Markscheme

attempt to substitute $x = 2$ into their $g(x)$ OR

complete the square on their $g(x)$ (may be seen in part (d)) **(M1)**

$$y = -\frac{25}{4} \quad \mathbf{A1}$$

[2 marks]

(f) Find the product of the solutions of the equation $f(x) = g(x)$.

[4]

Markscheme

$$\frac{4x+2}{x-2} = \left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) \text{ OR } \frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$$

attempt to form a cubic equation **(M1)**

EITHER

$$4x + 2 = (x - 2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) \text{ OR } 4x + 2 = \left(x^2 - 4x - \frac{9}{4}\right)(x - 2) \text{ OR}$$

$$(x - 2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) - 4x - 2 \text{ OR } (x - 2)\left(x^2 - 4x - \frac{9}{4}\right) - 4x - 2$$

$$x^3 + \dots + \frac{5}{2}(= 0) \text{ OR } 4x^3 + \dots + 10(= 0) \quad \mathbf{(A1)(A1)}$$

Note: Award **(A1)** for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

$$\text{product of roots} = \left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right) \text{ OR } \left(\frac{(-1)^3 \times 10}{4}\right)$$

$$= -\frac{5}{2} \quad \mathbf{A1}$$

OR

$$4\left(x + \frac{1}{2}\right) = (x - 2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$$

$$x = -\frac{1}{2} \quad \mathbf{(A1)}$$

$$\text{or } 4 = x^2 + \dots + 9 \Rightarrow x^2 + \dots + 5 = 0$$

product of roots of quadratic is 5 (A1)

product is therefore $-\frac{1}{2} \times 5$

$$= -\frac{5}{2} \quad \text{A1}$$

[4 marks]

11. [Maximum mark: 17]

24M.1.AHL.TZ1.11

Consider the polynomial $P(x) = 3x^3 + 5x^2 + x - 1$.

(a) Show that $(x + 1)$ is a factor of $P(x)$.

[2]

Markscheme

attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division (M1)

$$3(-1)^3 + 5(-1)^2 + (-1) - 1 = 0 \quad \text{OR} \quad \begin{array}{r|rrrr} & 3 & 5 & 1 & -1 \\ -1 & & -3 & -2 & 1 \\ \hline & 3 & 2 & -1 & 0 \end{array}$$

OR **Math input error** A1

[2 marks]

(b) Hence, express $P(x)$ as a product of three linear factors.

[3]

Markscheme

attempt to divide $P(x)$ by $(x + 1)$ e.g. using long division or synthetic division (M1)

$$P(x) = (x + 1)(3x^2 + 2x - 1) \quad \text{(A1)}$$

$$= (x + 1)(x + 1)(3x - 1) \left(= (x + 1)^2(3x - 1) \right) \quad \text{A1}$$

[3 marks]

Now consider the polynomial $Q(x) = (x + 1)(2x + 1)$.

- (c) Express $\frac{1}{Q(x)}$ in the form $\frac{A}{x+1} + \frac{B}{2x+1}$, where $A, B \in \mathbb{Z}$.

[3]

Markscheme

$$\frac{1}{(x+1)(2x+1)} \equiv \frac{A}{x+1} + \frac{B}{2x+1} \Rightarrow 1 \equiv A(2x+1) + B(x+1)$$

attempt to equate both coefficients OR substitute two values eg -1 and $-\frac{1}{2}$ (M1)

$$2A + B = 0 \text{ and } A + B = 1 \text{ OR } 1 = -A \text{ and } 1 = \frac{1}{2}B$$

$$A = -1 \text{ and } B = 2 \quad \mathbf{A1A1}$$

Note: Award **A1** for each value.

$$\frac{1}{(x+1)(2x+1)} = \frac{1}{x+1} + \frac{2}{2x+1}$$

[3 marks]

- (d) Hence, or otherwise, show that $\frac{1}{(x+1)Q(x)} = \frac{4}{2x+1} - \frac{2}{x+1} - \frac{1}{(x+1)^2}$.

[2]

Markscheme

$$\frac{1}{(x+1)(x+1)(2x+1)}$$

$$= \frac{1}{(x+1)} \left(-\frac{1}{x+1} + \frac{2}{2x+1} \right) \quad \mathbf{(A1)}$$

$$= -\frac{1}{(x+1)^2} + \frac{2}{(2x+1)(x+1)} \left(= -\frac{1}{(x+1)^2} + 2 \left(-\frac{1}{x+1} + \frac{2}{2x+1} \right) \right) \quad \mathbf{A1}$$

$$= \frac{4}{2x+1} - \frac{2}{x+1} - \frac{1}{(x+1)^2} \quad \mathbf{AG}$$

Note: Award **A1A0** for follow through from incorrect values in part (c).

[2 marks]

- (e) Hence, find $\int \frac{1}{(x+1)^2(2x+1)} dx$.

[4]

Markscheme

attempt to integrate at least one term in $\left(\frac{4}{2x+1} - \frac{2}{x+1} - \frac{1}{(x+1)^2}\right)$ (M1)

$$\int \left(\frac{4}{2x+1} - \frac{2}{x+1} - \frac{1}{(x+1)^2}\right) dx$$

$$= 2 \ln |2x + 1| - 2 \ln |x + 1| + \frac{1}{x+1} (+c) \quad \mathbf{A1A1}$$

Note: Award A1 for each correct term.

Award a maximum of **M1A1A0A1** if modulus signs are omitted.

Condone the absence of +c.

[4 marks]

Consider the function defined by $f(x) = \frac{P(x)}{(x+1)Q(x)}$, where $x \neq -1$, $x \neq -\frac{1}{2}$.

(f) Find

(f.i) $\lim_{x \rightarrow -1} f(x)$;

[2]

Markscheme

METHOD 1

attempt to cancel factors and substitute $x = -1$ (M1)

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim_{x \rightarrow -1} \left(\frac{3x-1}{2x+1}\right) = \frac{3(-1)-1}{2(-1)+1}$$

$$= 4 \quad \mathbf{A1}$$

METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x = -1$ (M1)

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x \rightarrow -1} \left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x \rightarrow -1} \left(\frac{18x+10}{12x+10}\right) = \frac{18(-1)+10}{12(-1)+10}$$

$$= 4 \quad \mathbf{A1}$$

[2 marks]

(f.ii) $\lim_{x \rightarrow \infty} f(x)$.

[1]

Markscheme

METHOD 1

attempt to consider coefficients of x^3 or divide all terms by x^3

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \left(\frac{3x^3 + \dots}{2x^3 + \dots} \right) \text{ or } \lim_{x \rightarrow \infty} \left(\frac{3 + \text{terms which tend to } 0}{2 + \text{terms which tend to } 0} \right) \\ &= \frac{3}{2} \quad A1\end{aligned}$$

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)} \right) = \lim_{x \rightarrow \infty} \left(\frac{3x-1}{2x+1} \right) \text{ or } \lim_{x \rightarrow \infty} \left(\frac{3 - \frac{1}{x}}{2 + \frac{1}{x}} \right) \\ &= \frac{3}{2} \quad A1\end{aligned}$$

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \left(\frac{3x^3 + 5x^2 + x - 1}{2x^3 + 5x^2 + 4x + 1} \right) = \lim_{x \rightarrow \infty} \left(\frac{9x^2 + 10x + 1}{6x^2 + 10x + 4} \right) = \lim_{x \rightarrow \infty} \left(\frac{18x + 10}{12x + 10} \right) = \lim_{x \rightarrow \infty} \left(\frac{18}{12} \right) \\ &= \frac{3}{2} \quad A1\end{aligned}$$

Note: If the *M1* has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

12. [Maximum mark: 20]

24M.1.AHL.TZ1.12

Consider $\Phi = (a + bi)^3$, where $a, b \in \mathbb{R}$.

(a) In terms of a and b , find

(a.i) the real part of Φ ;

[2]

Markscheme

attempt to expand the brackets or attempt to find modulus and argument of Φ (M1)

$$(a + bi)^3 = a^3 + 3a^2bi + 3a(bi)^2 + (bi)^3 \text{ OR}$$

$$\left(\sqrt{a^2 + b^2}\right)^3 \text{ cis} \left(3 \arctan \left(\frac{b}{a}\right)\right)$$

$$\text{real part is } a^3 - 3ab^2 \text{ OR } (a^2 + b^2)^{\frac{3}{2}} \cos \left(3 \arctan \left(\frac{b}{a}\right)\right)$$

[2 marks]

(a.ii) the imaginary part of Φ .

[1]

Markscheme

$$\text{imaginary part is } 3a^2b - b^3 \text{ OR } (a^2 + b^2)^{\frac{3}{2}} \sin \left(3 \arctan \left(\frac{b}{a}\right)\right) \quad A1$$

Note: Award (M1)A1A0 for $(a^3 - 3ab^2) + (3a^2b - b^3)i$ OR

$$(a^2 + b^2)^{\frac{3}{2}} \cos \left(3 \arctan \left(\frac{b}{a}\right)\right) + (a^2 + b^2)^{\frac{3}{2}} i \sin \left(3 \arctan \left(\frac{b}{a}\right)\right)$$

For (ii) condone $(3a^2b - b^3)i$ OR $(a^2 + b^2)^{\frac{3}{2}} i \sin \left(3 \arctan \left(\frac{b}{a}\right)\right)$

[1 mark]

(b) Hence, or otherwise, show that $(1 + \sqrt{3}i)^3 = -8$.

[2]

Markscheme

attempt to substitute $a = 1$ and $b = \sqrt{3}$ into their real or imaginary part found in (a) OR to expand the brackets OR to use polar form M1

$$(1 - 9) + (3\sqrt{3} - 3\sqrt{3})i \text{ OR}$$

$$(2\sqrt{3}i - 2)(1 + \sqrt{3}i) = 2\sqrt{3}i - 2 - 6 - 2\sqrt{3}i \text{ OR}$$

$$\left(2e^{\frac{i\pi}{3}}\right)^3 = 8e^{i\pi} \text{ OR } (2 \text{ cis } (60^\circ))^3 = 8 \text{ cis } (180^\circ) \quad A1$$

$$= -8 \quad \text{AG}$$

[2 marks]

The roots of the equation $z^3 = -8$ are u, v and w , where $u = 1 + \sqrt{3}i$ and $v \in \mathbb{R}$.

(c) Write down v and w , giving your answers in Cartesian form.

[2]

Markscheme

$$v = -2, \quad w = 1 - \sqrt{3}i \quad \text{A1A1}$$

Note: Award **A1A0** for $v = 1 - \sqrt{3}i$, $w = -2$ or if the labels v and w are not clearly specified or missing. Candidates may be awarded full **FT** marks for subsequent parts.

[2 marks]

On an Argand diagram, u, v and w are represented by the points **U**, **V** and **W** respectively.

(d) Find the area of the triangle **UVW**.

[3]

Markscheme

METHOD 1

triangle **UVW** has height $h = 3$ and base $b = 2\sqrt{3}$ (A1)

attempt to find area of triangle with their height and base (M1)

$$\text{area} = \frac{1}{2} \times 2\sqrt{3} \times 3$$

$$= 3\sqrt{3} \text{ (square units)} \quad \text{A1}$$

METHOD 2

triangle **UVW** has sides of length $\left(\sqrt{3^2 + (\sqrt{3})^2} = \right) \sqrt{12}$ (A1)

attempt to find area of equilateral triangle with their side length (M1)

$$\begin{aligned} \text{area} &= \frac{1}{2} (\sqrt{12})^2 \sin \frac{\pi}{3} \text{ OR } \frac{1}{2} \sqrt{12}(3) \text{ OR } (\sqrt{12})^2 \times \frac{\sqrt{3}}{4} \\ &= 3\sqrt{3} \text{ (square units)} \quad \mathbf{A1} \end{aligned}$$

METHOD 3

$$\text{triangle UVO has sides of length } \left(\sqrt{1^2 + (\sqrt{3})^2} = \right) 2 \quad \mathbf{(A1)}$$

attempt to find area of three isosceles triangles with their side length and angle $\frac{2\pi}{3}$ $\mathbf{(M1)}$

$$\begin{aligned} \text{area} &= 3 \left(\frac{1}{2} (2)^2 \sin \frac{2\pi}{3} \right) \\ &= 3\sqrt{3} \text{ (square units)} \quad \mathbf{A1} \end{aligned}$$

[3 marks]

Each of the points U , V and W is rotated counter-clockwise (anticlockwise) about O through an angle of $\frac{\pi}{4}$ to form three new points U' , V' and W' . These points represent the complex numbers u' , v' and w' respectively.

- (e) Find u' , v' and w' , giving your answers in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$.

[4]

Markscheme

attempt to express u , v or w in the form $re^{i\theta}$ and multiply by $e^{\frac{\pi}{4}i}$ $\mathbf{(M1)}$

$$\left(u' = 2e^{\frac{\pi}{3}i} e^{\frac{\pi}{4}i} = \right) 2e^{\frac{7\pi}{12}i} \quad \mathbf{A1}$$

$$\left(v' = 2e^{\pi i} e^{\frac{\pi}{4}i} = 2e^{\frac{5\pi}{4}i} = \right) 2e^{-\frac{3\pi}{4}i} \quad \mathbf{A1}$$

$$\left(w' = 2e^{-\frac{\pi}{3}i} e^{\frac{\pi}{4}i} = \right) 2e^{-\frac{\pi}{12}i} \quad \mathbf{A1}$$

Note: These **A1** marks should be awarded independently and in any order.

[4 marks]

- (f) Given that u, v and w are the solutions of $z^3 = c + di$, where $c, d \in \mathbb{R}$, find the value of c and the value of d .

[3]

Markscheme

EITHER

attempt to find one of $(u)^3, (v)^3$ or $(w)^3$ (M1)

$$(u)^3 = \left(2e^{\frac{7\pi}{12}i}\right)^3 = 8e^{\frac{7\pi}{4}i} = 8e^{-\frac{\pi}{4}i} \text{ OR } (v)^3 = \left(2e^{-\frac{3\pi}{4}i}\right)^3 = 8e^{-\frac{\pi}{4}i} \text{ OR}$$

$$(w)^3 = \left(2e^{-\frac{\pi}{12}i}\right)^3 = 8e^{-\frac{\pi}{4}i} \quad (A1)$$

OR

attempt to find product of their three roots u, v and w (M1)

$$u \times v \times w = (c + di)$$

$$2e^{\frac{7\pi}{12}i} \times 2e^{-\frac{3\pi}{4}i} \times 2e^{-\frac{\pi}{12}i} = 8e^{-\frac{\pi}{4}i} \text{ OR } 2e^{\frac{7\pi}{12}i} \times 2e^{\frac{5\pi}{4}i} \times 2e^{-\frac{\pi}{12}i} = 8e^{-\frac{\pi}{4}i} \text{ (or equivalent)} \quad (A1)$$

OR

attempt to find $\left(ze^{\frac{\pi}{4}i}\right)^3$ for any z such that $z^3 = -8$ OR to rotate -8 by $\frac{3\pi}{4}$ (M1)

$$\left(\left(ze^{\frac{\pi}{4}i}\right)^3 = z^3e^{\frac{3\pi}{4}i} = -8e^{\frac{3\pi}{4}i} \text{ OR } 8e^{-\frac{\pi}{4}i} \text{ OR } 8 \text{ cis } (-45^\circ)\right) \quad (A1)$$

THEN

$$8e^{-\frac{\pi}{4}i} = 8\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{8}{\sqrt{2}}(1 - i)$$

$$= 4\sqrt{2} - 4\sqrt{2}i \quad A1$$

Note: Accept $c = \frac{8}{\sqrt{2}}, d = -\frac{8}{\sqrt{2}}$.

[3 marks]

It is given that u, v, w, u', v' and w' are all solutions of $z^n = \alpha$ for some $\alpha \in \mathbb{C}$, where $n \in \mathbb{N}$.

(g) Find the smallest positive value of n .

[3]

Markscheme

METHOD 1

attempt to write the arguments of u, v, w, u', v' and w' over a common denominator OR to write the arguments in degrees (M1)

$$\frac{-9\pi}{12}, \frac{-4\pi}{12}, \frac{-\pi}{12}, \frac{4\pi}{12}, \frac{7\pi}{12}, \frac{12\pi}{12} \text{ OR } -135^\circ, -60^\circ, -15^\circ, 60^\circ, 105^\circ, 180^\circ$$

THEN

arguments of u, v, w, u', v' and w' differ by $\frac{3\pi}{12}$ and $\frac{5\pi}{12}$ OR 45° and 75°

so arguments of polygon vertices differ by $\frac{\pi}{12}$ or 15° (A1)

$$n = 24 \quad \text{A1}$$

METHOD 2

Let $z = r \operatorname{cis} \theta \Rightarrow z^n = r^n \operatorname{cis} (n\theta) = r^n \operatorname{cis} (n\theta)$, where θ is the argument of u, v, w, u', v' and w' .

recognition to find $n\theta$ where $n = 6, 12, 18, \dots$ and $\theta = -\frac{3\pi}{4}, -\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \pi$ (M1)

when $n = 6 \Rightarrow (n\theta =) -\frac{9\pi}{2}, -2\pi, -\frac{\pi}{2}, 2\pi, \frac{7\pi}{2}, 6\pi$

when $n = 12 \Rightarrow (n\theta =) -9\pi, -4\pi, -\pi, 4\pi, 7\pi, 12\pi$ (which is not a multiple of 2π) (A1)

$$n = 24 \quad \text{A1}$$

[3 marks]