Paper 1 Mock [110 marks]

1. [Maximum mark: 7] 23M.1.AHL.TZ1.1

The function f is defined by $f(x) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}, x \neq 2$.

(a) Find the zero of $f(x)$. [2]

Markscheme

recognizing $f(x) = 0$ ***(M1)***

$x = -1$ ***A1***

[2 marks]

(b) For the graph of $y = f(x)$, write down the equation of

(b.i) the vertical asymptote; [1]

Markscheme

$x = 2$ (must be an equation with x) ***A1***

[1 mark]

(b.ii) the horizontal asymptote. [1]

Markscheme

$y = \frac{7}{2}$ (must be an equation with y) ***A1***

[1 mark]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

Markscheme

EITHER

interchanging x and y ***(M1)***

$2xy - 4x = 7y + 7$

correct working with y terms on the same side: $2xy - 7y = 4x + 7$ ***(A1)***

OR

$2yx - 4y = 7x + 7$

correct working with x terms on the same side: $2yx - 7x = 4y + 7$ ***(A1)***

interchanging x and y OR making x the subject $x = \frac{4y+7}{2y-7}$ ***(M1)***

THEN

$f^{-1}(x) = \frac{4x+7}{2x-7}$ (or equivalent) $\left(x \neq \frac{7}{2}\right)$ ***A1***

[3 marks]

2. [Maximum mark: 6] 23M.1.AHL.TZ1.2

On a Monday at an amusement park, a sample of 40 visitors was randomly selected as they were leaving the park. They were asked how many times that day they had been on a ride called *The Dragon*. This information is summarized in the following frequency table.

Number of times on *The Dragon*	**Frequency**
0	6
1	16
2	13
3	2
4	3

It can be assumed that this sample is representative of all visitors to the park for the following day.

(a) For the following day, Tuesday, estimate

(a.i) the probability that a randomly selected visitor will ride *The Dragon*; [2]

Markscheme

summing frequencies of riders or finding complement ***(M1)***

probability$= \frac{34}{40}$ ***A1***

[2 marks]

(a.ii) the expected number of times a visitor will ride *The Dragon*. [2]

Markscheme

attempt to find expected value

$\frac{16}{40} + \left(2 \times \frac{13}{40}\right) + \left(3 \times \frac{2}{40}\right) + \left(4 \times \frac{3}{40}\right)$ ***(M1)***

$\frac{60}{40} (= 1.5)$ ***A1***

[2 marks]

It is known that 1000 visitors will attend the amusement park on Tuesday. *The Dragon* can carry a maximum of 10 people each time it runs.

(b) Estimate the minimum number of times *The Dragon* must run to satisfy demand. [2]

Markscheme

evidence of **their** rides/visitor $\times 1000$ or $\div 10$ ***(M1)***

1500 OR 0.15

150 (times) ***A1***

[2 marks]

3. [Maximum mark: 6] 23M.1.AHL.TZ1.3

Solve $\cos 2x = \sin x$, where $-\pi \leq x \leq \pi$. [6]

Markscheme

$1 - 2\sin^2 x = \sin x$ ***A1***

$2\sin^2 x + \sin x - 1 = 0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x - 1)(\sin x + 1)$ OR $\frac{-1 \pm \sqrt{1-4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x = \frac{1}{2}$ OR $\sin x = -1$

any correct solution from $\sin x = -1$ ***A1***

any correct solution from $\sin x = \frac{1}{2}$ ***A1***

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1***

Note: If no working shown, award no marks for a final value(s).

Award ***A0*** for $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ if additional values also given.

[6 marks]

4. [Maximum mark: 6] 23M.1.AHL.TZ1.4

Find the range of possible values of k such that $e^{2x} + \ln k = 3e^x$ has at least one real solution. [6]

Markscheme

recognition of quadratic in e^x ***(M1)***

$(e^x)^2 - 3e^x + \ln k \left(= 0\right)$ OR $A^2 - 3A + \ln k (= 0)$

recognizing discriminant ≥ 0 (seen anywhere) ***(M1)***

$(-3)^2 - 4\left(1\right)\left(\ln k\right)$ OR $9 - 4\ln k$ ***(A1)***

$\ln k \leq \frac{9}{4}$ ***(A1)***

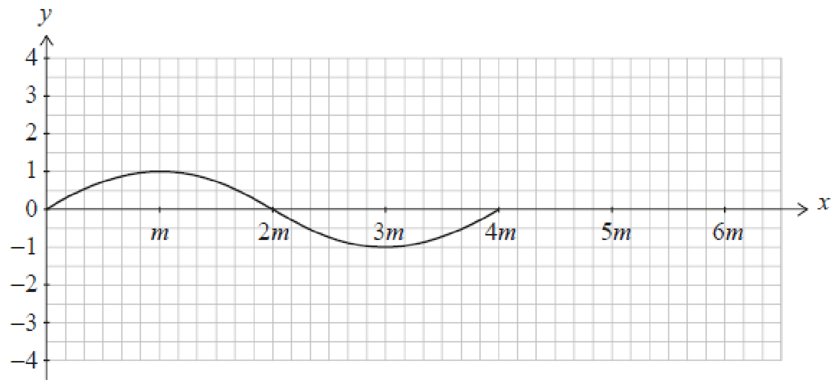
$e^{9/4}$ (seen anywhere) ***A1***

$0 < k \leq e^{9/4}$ ***A1***

[6 marks]

5. [Maximum mark: 6] 23M.1.AHL.TZ1.5

The function f is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of f for $0 \leq x \leq 4m$, where x is in radians. There are x-intercepts at $x = 0, 2m$ and $4m$.



(a) Find an expression for m in terms of q. [2]

Markscheme

recognition that period is $4m$ OR substitution of a point on f (except the origin) ***(M1)***

$4m = \frac{2\pi}{q}$ OR $1 = \sin qm$

$m = \frac{\pi}{2q}$ ***A1***

[2 marks]

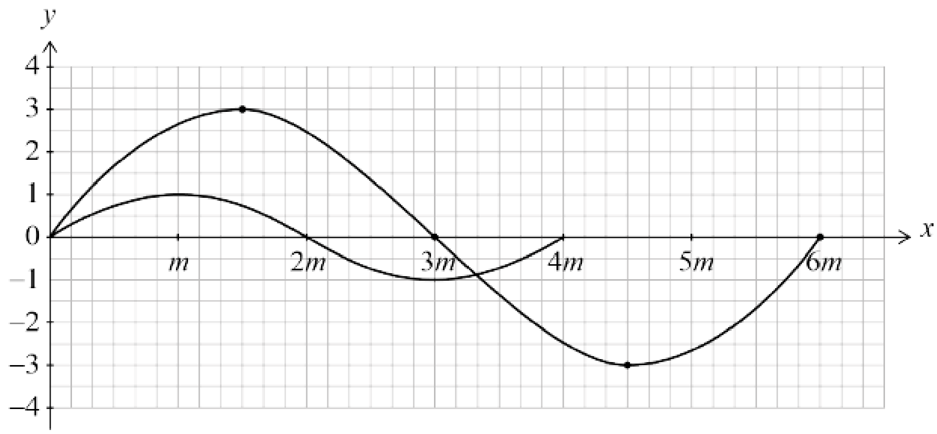
The function g is defined by $g\left(x\right) = 3 \sin \frac{2qx}{3}$, for $0 \leq x \leq 6m$.

(b) On the axes above, sketch the graph of g. [4]

Markscheme

horizontal scale factor is $\frac{3}{2}$ (seen anywhere) ***(A1)***

Note: This ***(A1)*** may be earned by seeing a period of $6m$, half period of $3m$ or the correct x-coordinate of the maximum/minimum point.



A1A1A1

Note:
Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
A1 for correct amplitude.
A1 for correct domain.
A1 for correct max and min points **and** correct x-intercepts.

[4 marks]

6. [Maximum mark: 5] 23M.1.AHL.TZ1.6

The side lengths, $x\ \mathrm{cm}$, of an equilateral triangle are increasing at a rate of $4\ \mathrm{cm\ s^{-1}}$.

Find the rate at which the area of the triangle, $A\ \mathrm{cm}^2$, is increasing when the side lengths are $5\sqrt{3}\ \mathrm{cm}$.

[5]

Markscheme

$A = \frac{1}{2}x^2 \sin\frac{\pi}{3}$ OR $A = \frac{1}{2}x^2 \sin 60\,^\circ$ OR triangle height $h = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}\ \left(= \frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$= \frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(= \frac{\sqrt{3}}{4}x^2\right)$ ***A1***

Note: Award ***A1*** for $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\sqrt{3}}{4} \times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} \times \sin\frac{\pi}{3} \times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ ***(A1)***

$= \frac{2\sqrt{3}}{4} \times 5\sqrt{3} \times 4$

$\frac{\mathrm{d}A}{\mathrm{d}t} = 30\left(\mathrm{cm}^2\ \mathrm{s}^{-1}\right)$ ***A1***

Note: Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

7. [Maximum mark: 6] 23M.1.AHL.TZ1.7

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$. [6]

Markscheme

METHOD 1

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

Note: Award ***A1A1*** for $P(3\mathrm{i})$ and $P(-3\mathrm{i}) = 0$ seen in their working.

Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\mathrm{i}) = 0$ or $P(-3\mathrm{i}) = 0$ ***(M1)***

$4m - 3m\mathrm{i} + \frac{36}{m}(3\mathrm{i})^2 - (3\mathrm{i})^3 = 0$

$4m - 3m\mathrm{i} - \frac{36}{m}(-9) + 27\mathrm{i} = 0$

attempt to equate the real or imaginary parts *(M1)*

$27-3m=0$ OR $9\times\frac{36}{m}=4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ *(M1)*

Note: Accept sum of three roots set to $-\frac{36}{m}$.

Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$3\mathrm{i}-3\mathrm{i}+r=\frac{36}{m}\left(\Rightarrow r=\frac{36}{m}\right)$

substitute their r into product of roots *(M1)*

$(3\mathrm{i})(-3\mathrm{i})\left(\frac{36}{m}\right)=4m$ OR $\left(z^2+9\right)\left(\frac{36}{m}-z\right)$

$9\times\frac{36}{m}=4m$ OR $\frac{4m}{9}=\frac{36}{m}$

OR

attempt to equate product of three roots to $4m$ *(M1)*

Note: Accept product of three roots set to $-4m$.

Award ***M0*** for stating product of roots is $\pm4m$.

$(3\mathrm{i})(-3\mathrm{i})\times r=4m\left(\Rightarrow r=\frac{4m}{9}\right)$

substitute their r into sum of roots *(M1)*

$3\mathrm{i}-3\mathrm{i}+\frac{4m}{9}=\frac{36}{m}$ OR $\left(z^2+9\right)\left(\frac{4m}{9}-z\right)$

$\frac{4m}{9}=\frac{36}{m}$

THEN

$m=9$ *(A1)*

third root is 4 ***A1***

METHOD 2

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

recognition that the other factor is $(z+3\mathrm{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(r-z)$ OR
$(z-3\mathrm{i})(z+3\mathrm{i})=z^2+9 \Rightarrow P(z)=(z^2+9)\left(\frac{4m}{9}-z\right)$

Note: Accept any attempt at long division of $P(z)$ by z^2+9.

Award ***M0*** for stating other factor is $(z+3\mathrm{i})$ or obtaining z^2+9 with no further working.

Attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

$m=9$ ***(A1)***

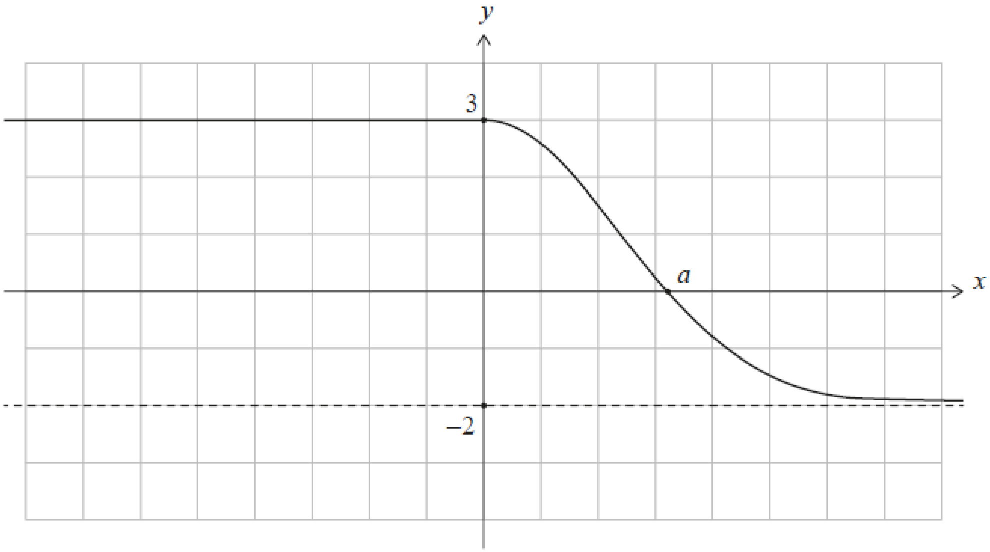
third root is 4 ***A1***

Note: Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

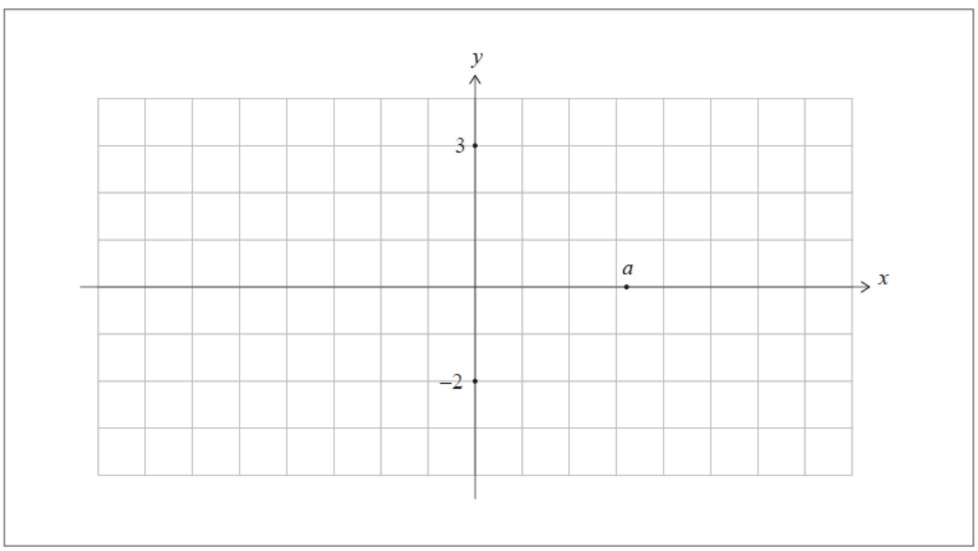
8. [Maximum mark: 7] 23M.1.AHL.TZ1.8

Part of the graph of a function, f, is shown in the following diagram. The graph of $y=f(x)$ has a y-intercept at $(0,\ 3)$, an x-intercept at $(a,\ 0)$ and a horizontal asymptote $y=-2$.



Consider the function $g(x) = |f(|x|)|$.

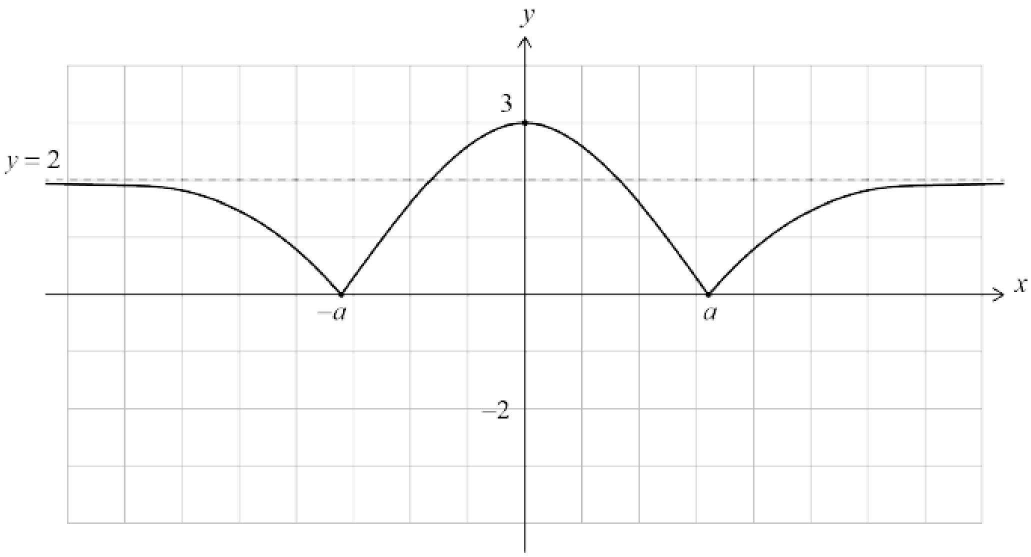
(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote.



[4]

Markscheme

attempt to reflect f in the x OR y axis ***(M1)***



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:

A1 for correct asymptotic behaviour at $y = 2$ (either side)

A1 for correctly reflected RHS of the graph in the y-axis with smooth maximum at $(0,\ 3)$.

A1 for labelled x-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

[4 marks]

(b) Find the possible values of k such that $\left(g(x)\right)^2 = k$ has exactly two solutions.

[3]

Markscheme

$k = 0$ ***A1***

$4 \le k < 9$ ***A2***

Note: If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.

Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the y-axis.

[3 marks]

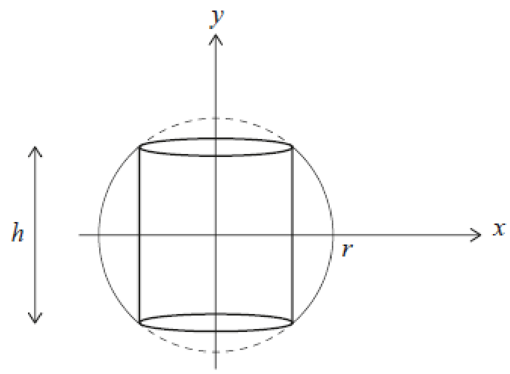
9. [Maximum mark: 7] 23M.1.AHL.TZ1.9

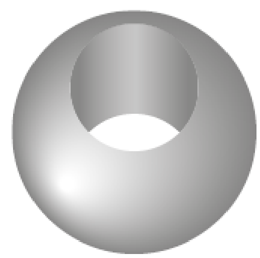
The function f is defined by $f\left(y\right) = \sqrt{r^2 - y^2}$ for $-r \leq y \leq r$.

The region enclosed by the graph of $x = f(y)$ and the y-axis is rotated by 360° about the y-axis to form a solid sphere. The sphere is drilled through along the y-axis, creating a cylindrical hole. The resulting spherical ring has height, h.

This information is shown in the following diagrams.

diagram not to scale





The spherical ring has a volume of π cubic units. Find the value of h. [7]

Markscheme

METHOD 1 (subtracting volumes)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $= \pi \int \left(r^2 - y^2\right) \mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2h}{2} - \frac{h^3}{24}$

recognition that the volume of the ring is $\pi \int (r^2 - y^2)\,\mathrm{d}y - \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi \int \left(r^2 - y^2\right)\mathrm{d}y - \pi\left(r^2 - \frac{h^2}{4}\right)h$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{r^2h}{2} - \frac{h^3}{24}\right) - \pi r^2 h + \frac{\pi h^3}{4} = \pi$ OR $\frac{h^3}{4} - \frac{h^3}{12} = 1$ (or equivalent)

OR

recognition that the volume of the ring is $\pi \int \left(\left(r^2 - y^2\right) - \left(r^2 - \frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$\frac{h^2}{4}y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{h^3}{8} - \frac{h^3}{24}$

correct equation ***(A1)***

$2\pi\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = \pi$ OR $2\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = 1$ (or equivalent)

THEN

$h^3 = \sqrt[3]{6}$ ***A1***

METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $= \pi \int \left(r^2 - y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3} - \frac{r^2h}{2} + \frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi \int \left(r^2 - y^2\right)\mathrm{d}y + \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi\left(r^2-\frac{h^2}{4}\right)h\left(=\frac{4}{3}\pi r^3-\pi\right)$ (or equivalent)

correct equation *(A1)*

$2\pi\left(\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}\right)+\pi r^2h-\frac{\pi h^3}{4}=\frac{4}{3}\pi r^3-\pi$ OR $\frac{h^3}{12}-\frac{h^3}{4}=-1$ (or equivalent)

$h=\sqrt[3]{6}$ ***A1***

METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) *(A1)*

attempt to use shells method *(M1)*

$2\pi\int x\sqrt{r^2-x^2}\mathrm{d}x$

correct limits r and $\sqrt{r^2-\frac{h^2}{4}}$ (seen anywhere) *(A1)*

correct integration ***A1***

$-\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression *(M1)*

$-\frac{1}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$

correct equation *(A1)*

$2\times\frac{-2\pi}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)=\pi$ OR $2\left(\frac{2\pi}{3}\times\frac{h^3}{8}\right)=\pi$

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

10. [Maximum mark: 14] 23M.1.AHL.TZ1.10

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n=n^2+4n$.

(a.i) Find the sum of the first five terms. [2]

Markscheme

recognition that $n = 5$ *(M1)*

$S_5 = 45$ *A1*

[2 marks]

(a.ii) Given that $S_6 = 60$, find u_6. [2]

Markscheme

METHOD 1

recognition that $S_5 + u_6 = S_6$ *(M1)*

$u_6 = 15$ *A1*

METHOD 2

recognition that $60 = \frac{6}{2}\left(S_1 + u_6\right)$ *(M1)*

$60 = 3(5 + u_6)$

$u_6 = 15$ *A1*

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ *(M1)*

$u_6 = 15$ *A1*

[2 marks]

(b) Find u_1. [2]

Markscheme

recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 *(M1)*

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}(U_1 + 15)$

$u_1 = 5$ ***A1***

[2 marks]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Markscheme

EITHER

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - (n^2 - 2n + 1 + 4n - 4)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}(5 + u_n)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n - 1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

Markscheme

recognition that $v_2 r^2 = v_4$ OR $(v_3)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) Given that $v_{99} < 0$, find v_5. [2]

Markscheme

recognition that r is negative ***(M1)***

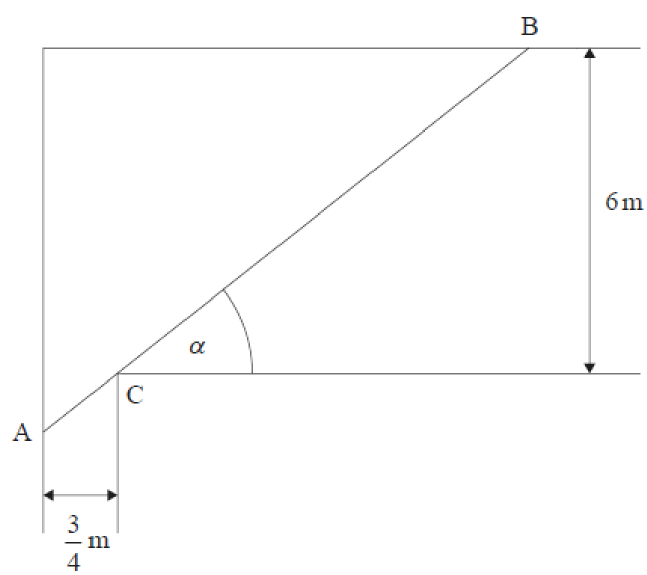
$v_5 = -15\sqrt{3}\ \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

11. [Maximum mark: 19] 23M.1.AHL.TZ1.11

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with $\frac{3}{4}$ m width is perpendicular to a room of width 6 m. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that $[\text{AB}]$ makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\,\alpha + 6\,\text{cosec}\,\alpha$. [2]

Markscheme

$L = \text{AC} + \text{CB}$

$\frac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\,\alpha\left(\Rightarrow \text{AC} = \frac{\frac{3}{4}}{\cos\,\alpha} \Rightarrow \text{AC} = \frac{3}{4}\sec\,\alpha\right)$ ***A1***

$\frac{6}{\text{CB}} = \sin\,\alpha\left(\Rightarrow \text{CB} = \frac{6}{\sin\,\alpha} \Rightarrow \text{CB} = 6\,\text{cosec}\,\alpha\right)$ ***A1***

so $L = \frac{3}{4}\sec\,\alpha + 6\,\text{cosec}\,\alpha$ ***AG***

[2 marks]

(b.i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$. [1]

Markscheme

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{3}{4}\sec\,\alpha\,\tan\,\alpha - 6\,\text{cosec}\,\alpha\,\cot\,\alpha$ ***A1***

[1 mark]

(b.ii) When $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, show that $\alpha = \arctan\,2$. [4]

Markscheme

attempt to write $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$ in terms of $\sin\,\alpha$, $\cos\,\alpha$ or $\tan\,\alpha$ (may be seen in (i)) ***(M1)***

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{\frac{3}{4}\sin\,\alpha}{\cos^2\,\alpha} - \frac{6\,\cos\,\alpha}{\sin^2\,\alpha}$ OR $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{\frac{3}{4}\tan\,\alpha}{\cos\,\alpha} - \frac{6}{\sin\,\alpha\,\cos\,\alpha}\left(= \frac{\frac{3}{4}\tan^3\,\alpha - 6}{\cos\,\alpha\,\tan^2\,\alpha}\right)$

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \frac{3}{4}\sin^3\,\alpha - 6\cos^3\,\alpha = 0$ OR $\frac{3}{4}\tan^3\,\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\,\alpha = 8$ ***A1***

$\tan\,\alpha = 2$ ***A1***

$\alpha = \arctan\,2$ ***AG***

[4 marks]

(c.i) Find $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$. [3]

Markscheme

attempt to use product rule (at least once) ***(M1)***

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$.

Allow unsimplified correct answer.

$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\alpha\tan^2\alpha + \frac{3}{4}\sec^3\alpha + 6\operatorname{cosec}\alpha\cot^2\alpha + 6\operatorname{cosec}^3\alpha\right)$

[3 marks]

(c.ii) When $\alpha = \arctan 2$, show that $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5}$. [4]

Markscheme

attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha = \arctan 2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\alpha = \sqrt{5}$ OR $\operatorname{cosec}\alpha = \frac{\sqrt{5}}{2}$ OR $\cot\alpha = \frac{1}{2}$ OR $\cos\alpha = \frac{1}{\sqrt{5}}$ OR $\sin\alpha = \frac{2}{\sqrt{5}}$

Note: ***M1A1*** may be seen in part (d).

$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3$ (or equivalent) ***A2***

$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$

Note: Award ***A1*** for only two or three correct terms.

Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$\frac{d^2L}{d\alpha^2}\frac{45}{4}\sqrt{5}$ ***AG***

[4 marks]

(d.i) Hence, justify that L is a minimum when $\alpha = \arctan 2$. [1]

Markscheme

$\frac{d^2L}{d\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{dL}{d\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

[1 mark]

(d.ii) Determine this minimum value of L. [2]

Markscheme

$(L_{\text{min}} =)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

Two people need to carry a pole of length $11.25\,\text{m}$ from the passageway into the room. It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]

Markscheme

$\left(11.25 =\right) \frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

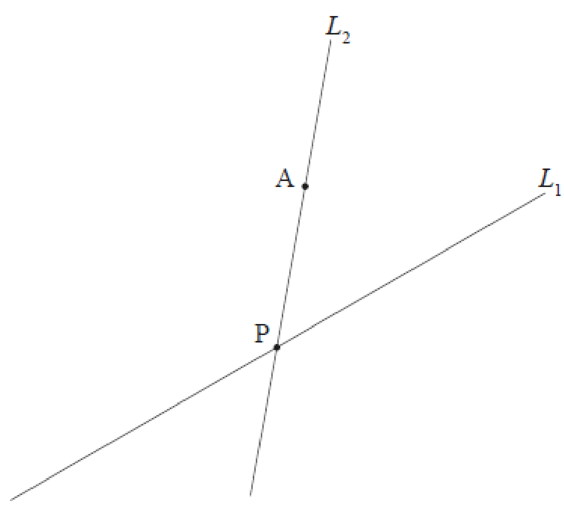
Note: Do not award ***R0A1.***

[2 marks]

12. [Maximum mark: 21] 23M.1.AHL.TZ1.12

Two lines, L_1 and L_2, intersect at point P. Point $\mathrm{A}(2t, 8, 3)$, where $t > 0$, lies on L_2. This is shown in the following diagram.

not to scale



The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of L_1 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\overrightarrow{\mathrm{PA}} = \begin{pmatrix} 2t \\ 0 \\ 3+t \end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$.

[4]

Markscheme

$2t + 1 \times 0 + 0 \times (3 + t) \ (= 2t)$ (seen anywhere) ***(A1)*** one correct magnitude $\sqrt{1^2+1^2+0^2}$, $\sqrt{(2t)^2+(3+t)^2}$ ***(A1)***

correct substitution of their magnitudes and scalar product ***M1***

$2t = \sqrt{2} \times \sqrt{(2t)^2+(3+t)^2} \times \cos\frac{\pi}{3}$ OR $\cos\frac{\pi}{3} = \frac{2t}{\sqrt{2}\times\sqrt{5t^2+6t+9}}$

$4t = \sqrt{2(4t^2+9+6t+t^2)}$ OR $\frac{1}{2} = \frac{2t}{\sqrt{2(5t^2+6t+9)}}$ (or equivalent) ***A1***

$4t = \sqrt{10t^2+12t+18}$ ***AG***

[4 marks]

(b) Find the value of t. [4]

Markscheme

correct quadratic equation ***A1***

$16t^2 = 10t^2+12t+18$, $6t^2-12t-18=0$, $t^2-2t-3=0$

valid attempt to solve their quadratic set $=0$ ***(M1)***

$(t+1)(t-3)$ OR $\frac{12\pm\sqrt{(-12)^2-4\times6\times(-18)}}{12}$ OR $(t-1)^2-4$ ***(A1)***

$t=3$ ***A1***

Note: Award ***A0*** if additional answer(s) given.

[4 marks]

(c) Hence or otherwise, find the shortest distance from A to L_1.

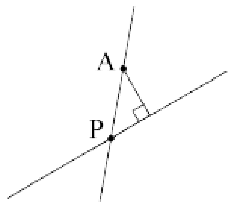
[4]

Markscheme

METHOD 1

recognizing shortest distance from A is perpendicular to L_1 ***(M1)***

OR



$$\sin\frac{\pi}{3} = \frac{x}{\left|\overrightarrow{\mathrm{PA}}\right|}$$

$\left|\overrightarrow{\mathrm{PA}}\right| = \sqrt{6^2+6^2} \ \left(= \sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

$\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{72}}$ ***(A1)***

$x = \frac{\sqrt{216}}{2} \ \left(= \sqrt{54},\ 3\sqrt{6}\right)$

shortest distance is $\frac{\sqrt{216}}{2} \ \left(= \sqrt{54},\ 3\sqrt{6}\right)$ ***A1***

METHOD 2

recognition that the distance required is $\frac{\left|\boldsymbol{v}+\overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}6\\0\\6\end{pmatrix}\right|$ ***(A1)***

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}6\\-6\\-6\end{pmatrix}\right|$ ***(A1)***

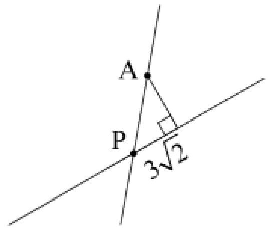
shortest distance is $\sqrt{54} \ \left(= 3\sqrt{6}\right)$ ***A1***

METHOD 3

recognition that the base of the triangle is $\frac{\left|\boldsymbol{v}\cdot\overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}6\\0\\6\end{pmatrix}\right|$

$= \frac{6}{\sqrt{2}}\ \left(= 3\sqrt{2}\right)$ OR



(A1)

$\left|\overrightarrow{\text{PA}}\right| = \sqrt{6^2 + 6^2}\ \ \left(= \sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) *(A1)*

Note: The value of $\left|\overrightarrow{\text{PA}}\right| = \sqrt{6^2 + 6^2}$ may be seen as part of the working of their shortest distance,

$d = \sqrt{\left|\overrightarrow{\text{PA}}\right| - b^2} = \sqrt{\left(\sqrt{72}\right)^2 - \left(3\sqrt{2}\right)^2}$

shortest distance is $\sqrt{54}\ \left(= 3\sqrt{6}\right)$ *A1*

METHOD 4

Let B be a general point on $L_1\,(\lambda,\ 8+\lambda,\ -3)$ such that AB is perpendicular to L_1

attempt to find vector $\overrightarrow{\text{AB}}$ OR $\left|\overrightarrow{\text{AB}}\right|$ (the shortest distance from B to L_1) *(M1)*

$$\overrightarrow{\text{AB}} = \overrightarrow{\text{OB}} - \overrightarrow{\text{OA}} = \overrightarrow{\text{OP}} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \overrightarrow{\text{OA}}\left(= \begin{pmatrix}0\\8\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}6\\8\\3\end{pmatrix} = \overrightarrow{\text{AP}} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right)\quad (\lambda \in \mathbb{R})$$

$\overrightarrow{\text{AB}} = \begin{pmatrix}-6\\0\\-6\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ OR $\left|\overrightarrow{\text{AB}}\right| = \sqrt{(\lambda-6)^2 + (8+\lambda-8)^2 + (-3-3)^2}$ *A1*

$\left|\overrightarrow{\text{AB}}\right| = \sqrt{(\lambda-6)^2 + \lambda^2 + (-6)^2}\quad \left(= \sqrt{2\lambda^2 - 12\lambda + 72}\right)$

EITHER

$\frac{d}{d\lambda}\left(\left|\overrightarrow{AB}\right|^2\right) = 0 \Rightarrow 4\lambda - 12 = 0 \Rightarrow \lambda = 3$ ***A1***

OR

$\left|\overrightarrow{AB}\right| = \sqrt{2(\lambda - 3)^2 + 54}$ to obtain $\lambda = 3$ ***A1***

OR

$\begin{pmatrix} -6+\lambda \\ \lambda \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow -6 + \lambda + \lambda = 0 \Rightarrow \lambda = 3$ ***A1***

THEN

shortest distance is $\sqrt{54}$ $\left(= 3\sqrt{6}\right)$ ***A1***

[4 marks]

A plane, Π, contains L_1 and L_2.

(d) Find a normal vector to Π. [2]

Markscheme

attempt to find the vector product of two direction vectors ***(M1)***

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$

$\boldsymbol{n} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (or any scalar multiple of this) (accept $\boldsymbol{n} =< 1, \ -1, \ -1 >$ or equivalent) ***A1***

Note: Award ***A0*** for a final answer given in coordinate form.

[2 marks]

The base of a right cone lies in Π centred at A such that L_1 is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

Markscheme

substituting their x into volume formula and equating ***(M1)***

$\frac{1}{3}\pi\left(3\sqrt{6}\right)^2 h = 90\sqrt{3}\pi$

$h = 5\sqrt{3}$ (seen anywhere) ***A1***

recognition that the position vector of vertex is given by $\overrightarrow{\mathrm{OA}} + \mu\boldsymbol{n}$ OR $\overrightarrow{\mathrm{OA}} + h \times \widehat{\boldsymbol{n}}$ ***(M1)***

$\begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ OR $(6+\mu,\ 8-\mu,\ 3-\mu)$

EITHER

recognition that $\mu|\boldsymbol{n}| = h$ (where μ is a parameter) ***(M1)***

$\mu|\boldsymbol{n}| = 5\sqrt{3}$ OR $\sqrt{\mu^2 + (-\mu)^2 + (-\mu)^2} = 5\sqrt{3}$ OR $3\mu^2 = 75$ $\left(\Rightarrow \sqrt{3}\mu = 5\sqrt{3}\right)$

$\mu = \pm 5$ (accept $\mu = 5$) ***(A1)***

OR

attempt to find cone's height vector $h \times \widehat{\boldsymbol{n}}$ ***(M1)***

$\widehat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ ***(A1)***

$5\sqrt{3} \times \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

THEN

$$= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm 5 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \left(= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm \begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} \right)$$

vertex = $(11, 3, -2)$ and $(1, 13, 8)$ (accept position vectors) ***A1A1***

Note: Award a maximum of ***(M0)A0(M1)(M1)(A1)A1A1FT*** for

$\left(\frac{39}{4}, \frac{17}{4}, -\frac{3}{4}\right)$ and $\left(\frac{9}{4}, \frac{47}{4}, \frac{27}{4}\right)$ obtained using $x = \left|\overrightarrow{\mathrm{PA}}\right|$ from part (c).

[7 marks]

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