Paper 2 Mock Mock [110 marks]

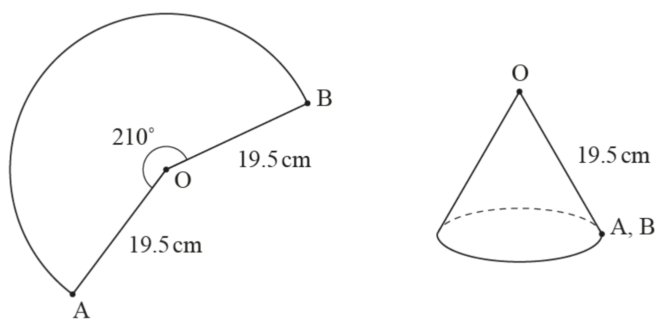
1. [Maximum mark: 6] 25M.2.AHL.TZ1.1

The points A and B lie on a circle, with centre O and radius 19.5 cm, such that $\mathrm{B\hat{O}A} = 210^\circ$.

A piece of paper is cut into the shape of the sector BOA.

A hollow cone with no base is constructed from the sector by joining the points A and B. The sector forms the curved surface of the cone.

This is shown in the following diagrams.



Find

(a) the area of the sector BOA; [3]

Markscheme

EITHER

$210^\circ = \frac{210\pi}{180}\left(= \frac{7\pi}{6} = 3.66519\ldots\right)$ radians ***(A1)***

attempt to use radian formula for area of sector ***(M1)***

area $= \frac{1}{2}(19.5)^2\left(\frac{7\pi}{6}\right)$

OR

attempt to use degree formula for area of sector ***(M1)***

area $= \frac{210}{360}\pi(19.5)^2$ ***(A1)***

THEN

area $= \frac{3549\pi}{16} = 696.844\ldots$

$= 697\left(= \frac{3549\pi}{16}\right)\ (\text{cm}^2)$ ***A1***

[3 marks]

(b) the radius of the cone. [3]

Markscheme

EITHER

arc length $= 19.5\left(\frac{7\pi}{6}\right)$ **OR** $= \frac{210}{360}(2\pi(19.5))\left(= \frac{91\pi}{4} = 71.4712\ldots\right)$ ***(A1)***

attempt to set $2\pi r$ equal to arc length ***(M1)***

$2\pi r = 71.4712\ldots$

OR

attempt to set πrl equal to their area from (a) ***(M1)***

$19.5\pi r = 696.844\ldots$ ***(A1)***

THEN

$r = 11.4\left(= \frac{91}{8} = 11.375\right)$ (cm) ***A1***

[3 marks]

2. [Maximum mark: 4] 25M.2.AHL.TZ1.2

Consider the function $f(x) = a\tan(2x) + b$, where $x \neq \frac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$ and $a,\ b \in \mathbb{R}$.

(a) Write down the period of f. [1]

Markscheme

period is $\frac{\pi}{2}(= 1.57079\ldots = 1.57)$ ***A1***

[1 mark]

The graph of $y = f(x)$ passes through the points $\left(\frac{\pi}{12},\ 5\right)$ and $\left(\frac{\pi}{3},\ 7\right)$.

(b) Find the value of a and the value of b. [3]

Markscheme

attempt to substitute $x = \frac{\pi}{12},\ f(x) = 5$ and $x = \frac{\pi}{3}, f(x) = 7$ to obtain two equations ***(M1)***

Note: accept work where x values have been converted into degrees

$a\tan\left(\frac{\pi}{6}\right)+b=5$ and $a\tan\left(\frac{2\pi}{3}\right)+b=7\left(\Rightarrow\frac{a}{\sqrt{3}}+b=5\text{ and }-a\sqrt{3}+b=7\right)$

$a=-\frac{\sqrt{3}}{2}(=-0.866025\ldots=-0.866)$ ***A1***

$b=\frac{11}{2}(=5.5)$ ***A1***

Note: These ***A1*** marks may be awarded independently.

[3 marks]

3. [Maximum mark: 6] 25M.2.AHL.TZ1.3

A population, P, has a rate of change modelled by $\frac{\mathrm{d}P}{\mathrm{d}t}=-104\,000\mathrm{e}^{-0.0145t}$, where t is the time measured in years since the **start** of 2022.

At the start of 2022, the population was 6.78×10^6.

Based on this model, find the predicted population at the start of 2026. [6]

Markscheme

METHOD 1

attempt to find change in population using a definite integral ***(M1)***

$t=4$ at the start of 2026 (seen anywhere) ***(A1)***

$\int_0^4 -104\,000\mathrm{e}^{-0.0145t}\,\mathrm{d}\,t$ ***(A1)***

$=-404165.8\ldots$ ***(A1)***

attempt to add initial population to their change in population from a definite integral ***(M1)***

population at the start of $2026=6.78\times10^6-404\,165.8\ldots$

$=6\,375\,834.1\ldots$

$=6\,380\,000\ \left(=6.38\times10^6\right)$ ***A1***

METHOD 2

attempt to find population using an indefinite integral ***(M1)***

$P=\int -104\,000\mathrm{e}^{-0.0145t}\,\mathrm{d}\,t$

$\frac{-104\,000\mathrm{e}^{-0.0145t}}{-0.0145}+c\left(=7\,172\,413.7\ldots\mathrm{e}^{-0.0145t}+c\right)$ ***(A1)*** attempt to substitute $t = 0,\ P = 6.78 \times 10^6$ into equation with c. ***(M1)***

$6.78 \times 10^6 = 7172413.7\ldots + c \Rightarrow c = -392413.7\ldots$

$P = 7172413.7\ldots \mathrm{e}^{-0.0145t} - 392413.7\ldots$ ***(A1)***

$t = 4$ at the start of 2026 (seen anywhere) ***(A1)***

population at the start of $2026 = 7172413.7\ldots \mathrm{e}^{-0.0145(4)} - 392413.7\ldots$

$= 6375834.1\ldots$

$= 6380000 (= 6.38 \times 10^6)$ ***A1***

[6 marks]

4. [Maximum mark: 8] 25M.2.AHL.TZ1.4

In a study, measurements for arm span, A cm, and foot length, F cm, are taken from a large group of adults.

For this group, the regression line of F on A is found to be $F = 0.335A - 32.6$, and the regression line of A on F is found to be $A = 2.89F + 99.3$. Each regression line passes through the mean point.

(a) By using an appropriate regression line, find an estimate of the arm span for an adult with a foot length of 19.8 cm. [2]

Markscheme

attempt to substitute $F = 19.8$ into the regression line for A on F ***(M1)***

$A = 2.89(19.8) + 99.3$

$= 156.522\,(\mathrm{cm})$

arm span $= 157\,(\mathrm{cm})$ ***A1***

Note: Award ***M0A0*** for choosing the wrong regression line to get $A = 156.417\ldots$ so $A = 156$.

[2 marks]

(b) For this group of adults, find the mean arm span and the mean foot length. [3]

Markscheme

recognition that the lines intersect at the mean point (may be seen on a sketch) ***(M1)***

$2.89F + 99.3 = \frac{F+32.6}{0.335}$ **OR** $0.335A - 32.6 = \frac{A-99.3}{2.89}$

$159.686\ldots$ **OR** $20.8948\ldots$

the mean arm span $= 160\,(\mathrm{cm})$, the mean foot length $= 20.9\,(\mathrm{cm})$ ***A1A1***

[3 marks]

The heights, $H\ \text{cm}$, of adults in the group can be modelled by a normal distribution with mean $163\ \text{cm}$ and standard deviation $\sigma\ \text{cm}$.

It is found that 88% of the group have a height between $153\ \text{cm}$ and $173\ \text{cm}$.

(c) Find the value of σ. [3]

Markscheme

METHOD 1

recognition of symmetry of interval around mean (may be seen on a sketch) ***(M1)***

$P(H < 153) = 0.06$ **OR** $P(H < 173) = 0.94$ **OR** equivalent

$\frac{153-163}{\sigma} = -1.55477\ldots$ **OR** $\frac{173-163}{\sigma} = -1.55477\ldots$ ***(A1)***

$\sigma = 6.43181\ldots$

$\sigma = 6.43\,(\text{cm})$ ***A1***

METHOD 2

attempt to find σ by equating an appropriate correct normal CDF function to 0.88 (or e.g. 0.06 or 0.94) ***(M1)***

$\sigma = 6.43181\ldots$

$\sigma = 6.43\,(\text{cm})$ ***A2***

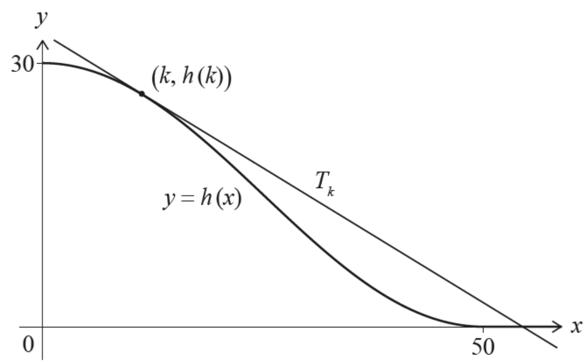
Note: Accept use of calculator notation eg $normcdf\,(153, 173, 163, \sigma) = 0.88$

[3 marks]

5. [Maximum mark: 6] 25M.2.AHL.TZ1.5

Consider the function $h(x) = 15\cos\left(\frac{\pi x}{50}\right) + 15$, where $0 \le x \le 50$.

The tangent, T_k, to the curve $y = h(x)$ at the point $(k,\ h(k))$ is shown on the following diagram.



(a) Find the gradient of T_k in terms of k. [3]

Markscheme

recognition of the need to differentiate *(M1)*

$h\prime(x)=\frac{\pi}{50}\left(-15\sin\left(\frac{\pi x}{50}\right)\right)\left(=-\frac{15\pi}{50}\sin\left(\frac{\pi x}{50}\right)=-\frac{3\pi}{10}\sin\left(\frac{\pi x}{50}\right)\right)$ ***A1A1***

$h\prime(k)=-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right)\left(=-\frac{3\pi}{10}\sin\left(\frac{\pi k}{50}\right)\right)$

Note: Award ***A1*** for $-15\sin\left(\frac{\pi k}{50}\right)$ and ***A1*** for factor of $\frac{\pi}{50}$.
Award ***A1A0*** for a correct expression with additional terms or additional factors.

[3 marks]

Consider the case where the angle between T_k and the x-axis is $\frac{\pi}{8}$ radians.

(b) Find the possible values of k. [3]

Markscheme

recognition that gradient of tangent $=-\tan\left(\frac{\pi}{8}\right)$ **OR** $\tan\left(\frac{7\pi}{8}\right)$ ***(M1)***

Note: Accept $\tan\left(\frac{\pi}{8}\right)$ **OR** $-\tan\left(\frac{7\pi}{8}\right)$ for the ***(M1)***

setting their $h\prime(k)$ equal to $-\tan\left(\frac{\pi}{8}\right)$ **OR** $\tan\left(\frac{7\pi}{8}\right)$ $(=-0.414213\ldots)$ ***(A1)***

$-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right)=-\tan\left(\frac{\pi}{8}\right)$ **OR** $-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right)=\tan\left(\frac{7\pi}{8}\right)$

$k=7.24211\ldots,\ k=42.7578\ldots$

$k = 7.24,\ k = 42.8$ ***A1***

[3 marks]

6. [Maximum mark: 7] 25M.2.AHL.TZ1.6

Consider the function $f(x) = 4\cot x + \sin x$, where $0 < x < \pi$.

(a.i) Write $f(x)$ in terms of $\sin x$ and $\cos x$. [1]

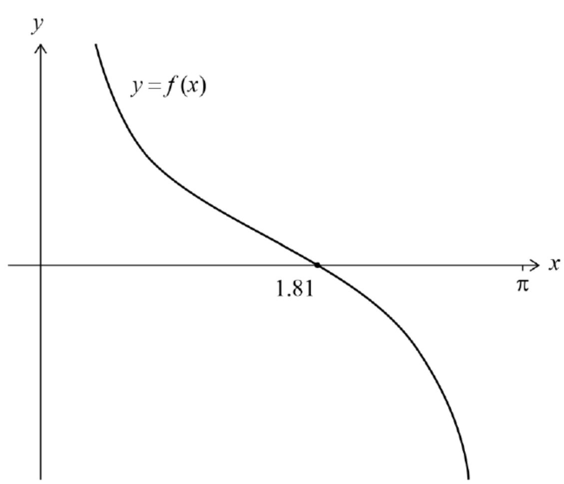
Markscheme

$\left(f(x) =\right)\frac{4\cos x}{\sin x} + \sin x$ ***A1***

[1 mark]

(a.ii) Hence or otherwise, sketch the graph of $y = f(x)$, showing the value of the x-intercept. [2]

Markscheme



correct domain and curvature ***A1***

Note: The correct domain may be implied by a single branch of the function. Condone the absence of π on the x-axis.

Award ***A0*** if the graph is not a function.

correct x-intercept at $x = 1.80911\ldots = 1.81$ ***A1***

Note: Accept coordinates $(1.81,\ 0)$. This mark is independent of the first ***A1***.

[2 marks]

(b) Find the value of $f^{-1}(2)$. [1]

Markscheme

$f^{-1}(2) = 1.31837...$

$= 1.32$ ***A1***

[1 mark]

It is given that $\sec\ \alpha = 1.5$, where $0 < \alpha < \pi$.

(c) Find the value of $f(\alpha)$. [3]

Markscheme

METHOD 1

EITHER

recognition that $\sec\ x = \frac{1}{\cos\ x}$ ***(M1)***

$\cos\ \alpha = \frac{2}{3}(= 0.666666...)$

$\alpha = \arccos\ \frac{2}{3}(= 0.841068...)$ ***(A1)***

OR

$\alpha = \text{arcsec}\ 1.5(= 0.841068...)$ ***(A2)***

THEN

$f(\alpha) = f(0.841068...) = 4.32306...$

$= 4.32\left(= \frac{29\sqrt{5}}{15}\right)$ ***A1***

METHOD 2

recognition that $\sec\ x = \frac{1}{\cos\ x}$ ***(M1)***

$\cos\ \alpha = \frac{2}{3},\ \sin\ \alpha = \frac{\sqrt{5}}{3}\left(\cot\ x = \frac{2}{\sqrt{5}}\right)$ ***(A1)***

$f(\alpha) = \frac{4\left(\frac{2}{3}\right)}{\frac{\sqrt{5}}{3}} + \frac{\sqrt{5}}{3}\left(= 4\left(\frac{2}{\sqrt{5}}\right) + \frac{\sqrt{5}}{3}\right)$

$= \frac{29\sqrt{5}}{15}(= 4.32306... = 4.32)$ ***A1***

[3 marks]

7. [Maximum mark: 5] 25M.2.AHL.TZ1.7

At $09:00$ a helicopter is located at a point $(10,\ 3,\ 0.5)$ relative to a point O on horizontal ground. The x-direction is due east, the y-direction is due north and the z-direction is vertically upwards.

All distances are measured in kilometres.

The helicopter is flying at a constant height.

The helicopter's position relative to the point O is given by $\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 0.5 \end{pmatrix} + 4t\begin{pmatrix} 10 \\ -25 \\ 0 \end{pmatrix}$, where t represents the time in hours since $09:00$.

(a) Find the speed of the helicopter. [2]

Markscheme

recognition that the speed is the magnitude of the velocity ***(M1)***

speed $= 4\sqrt{10^2 + (-25)^2}$

$= 107.703\ldots$

$= 108 \ \left(= 20\sqrt{29}\right)$ (km/h) ***A1***

[2 marks]

At $10:00$ the helicopter begins to descend.

During descent the helicopter's vertical height decreases at a constant rate of $16\ \mathrm{km\ h^{-1}}$ and its horizontal velocity remains unchanged.

The angle of descent, β, is defined as the angle between the helicopter's direction of travel and the horizontal.

(b) Find β, giving your answer in degrees. [3]

Markscheme

METHOD 1

attempt to use right-angled triangle with the horizontal speed found in (a) ***(M1)***

$\tan\beta = \frac{16}{20\sqrt{29}} \left(= \frac{4}{5\sqrt{29}}\right)$ ***(A1)***

$\beta = 8.44984(^\circ)$

$\beta = 8.45(^\circ)$ ***A1***

Note: Award ***M1A1A0*** for answer of 0.147 radians.

METHOD 2

attempt to find angle between $\begin{pmatrix}10\\-25\\-4\end{pmatrix}$ and $\begin{pmatrix}10\\-25\\0\end{pmatrix}$ ***(M1)***

$$\cos\beta=\frac{\begin{pmatrix}10\\-25\\-4\end{pmatrix}\cdot\begin{pmatrix}10\\-25\\0\end{pmatrix}}{\sqrt{10^2+(-25)^2+(-4)^2}\sqrt{10^2+(-25)^2}}\left(=\frac{725}{\sqrt{741}\sqrt{725}}=0.989144\ldots\right)$$ **OR**

$$\sin\beta=\frac{\left|\begin{pmatrix}10\\-25\\-4\end{pmatrix}\times\begin{pmatrix}10\\-25\\0\end{pmatrix}\right|}{\sqrt{10^2+(-25)^2+(-4)^2}\sqrt{10^2+(-25)^2}}=\frac{\left|\begin{pmatrix}-100\\-40\\0\end{pmatrix}\right|}{\sqrt{741}\sqrt{725}}\left(=\frac{4}{\sqrt{741}}=0.146943\ldots\right)$$ ***(A1)***

$\beta=8.44984(^\circ)$

$\beta=8.45(^\circ)$ ***A1***

METHOD 3

attempt to find angle between $\begin{pmatrix}10\\-25\\-4\end{pmatrix}$ and a plane parallel to $z=0$ ***(M1)***

$$\sin\beta=\frac{\left|\begin{pmatrix}10\\-25\\-4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\0\end{pmatrix}\right|}{\sqrt{10^2+(-25)^2+(-4)^2}}\left(=\frac{4}{\sqrt{741}}=0.146943\ldots\right)$$ ***(A1)***

Note: This could also be written as $\cos(90^\circ-\beta)=\ldots$

$\beta=8.44984(^\circ)$

$\beta=8.45(^\circ)$ ***A1***

[3 marks]

8. [Maximum mark: 7] 25M.2.AHL.TZ1.8

Consider the functions $f,\ g$ and h defined as follows for $t\in\mathbb{R}$.

$f(t)=\sin(2t+1)$

$g(t) = \sin\,(2t + 3)$

$h(t) = f(t) + g(t)$

(a) Show that $h(t) = \mathrm{Im}\left(\mathrm{e}^{2ti}\left(\mathrm{e}^{i} + \mathrm{e}^{3i}\right)\right)$. [2]

Markscheme

$\mathrm{Im}\,\left(\mathrm{e}^{2ti}\left(\mathrm{e}^{i} + \mathrm{e}^{3i}\right)\right)$

$= \mathrm{Im}\left(\mathrm{e}^{(2t+1)i} + \mathrm{e}^{(2t+3)i}\right)$ ***A1***

Note: This ***A1*** is for clearly showing that the powers are added.
Accept alternative notation for the step of adding the arguments e.g.
$(\cos\,(2t) + i\,\sin\,(2t))(\cos\,(1) + i\,\sin\,(1)) = \cos\,(2t + 1) + i\,\sin\,(2t + 1)$

$= \sin\,(2t + 1) + \sin\,(2t + 3)$ ***A1***

$= h(t)$ ***AG***

Note: Accept argument in reverse

[2 marks]

(b) Write $\mathrm{e}^{i} + \mathrm{e}^{3i}$ in the form $r\mathrm{e}^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

Markscheme

$\mathrm{e}^{i} + \mathrm{e}^{3i} = 1.08060\ldots\mathrm{e}^{2i}$

$1.08\mathrm{e}^{2i}$ ***A1A1***

Note: Award ***A1*** for modulus ***A1*** for argument

$r = 1.08\,(= 2\,\cos\,(1))\,,\ \theta = 2$

[2 marks]

(c) Hence or otherwise, write $h(t)$ in the form $p\,\sin\,(2t + q)$, where $p > 0$ and $0 < q < 2\pi$. [3]

Markscheme

METHOD 1

attempt to use their answers to part (a) and (b) to write $h(t)$ as the imaginary part of a number in the form $r\mathrm{e}^{i\theta}$
(M1)

$h(t) = \mathrm{Im}\,\left(\mathrm{e}^{2ti}\left(1.08060\ldots\mathrm{e}^{2i}\right)\right)$

$= \mathrm{Im}\,\left(1.08060\ldots\mathrm{e}^{(2t+2)i}\right)$ ***(A1)***

$h(t) = 1.08060... \sin(2t + 2)$

$= 1.08 \sin(2t + 2)\ (= 2\cos(1)\sin(2t + 2))$ ***A1***

METHOD 2

Considering the graph of $h(t)$

amplitude $= 1.08060...$

$p = 1.08\ (= 2\cos(1))$ ***(A1)***

attempt to consider the horizontal shift of the graph of h ***(M1)***

first negative zero of graph is -1

$h(t) = 1.08060... \sin(2(t + 1))\ (= 1.08060... \sin(2t + 2))$

$= 1.08 \sin(2t + 2)\ (= 2\cos(1)\sin(2t + 2))$ ***A1***

[3 marks]

9. [Maximum mark: 6] 25M.2.AHL.TZ1.9

Consider the differential equation $\frac{dy}{dx} = \frac{2x}{x^2+y}$.

The solution curve passes through the point $(1,\ 0)$.

(a) Use Euler's method with a step value of 0.25 to estimate the value of y when $x = 2$. [3]

Markscheme

attempt to use Euler's method with a step of 0.25 ***(M1)***

$(x_{n+1} = x_n + 0.25),\ y_{n+1} = y_n + 0.25\left(\frac{2x_n}{{x_n}^2+y_n}\right)$

$(y_0 = 0)$

$y_1 = 0.5$

$y_2 = 0.803030...$ ***(A1)***

$y_3 = 1.04868...$

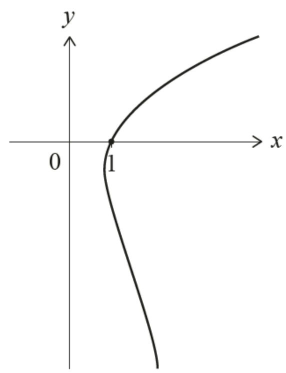
Note: Award ***(A1)*** for at least two correct intermediate values given to 3sf.

If $x = 2,\ y = 1.26152...$

$y = 1.26$ ***A1***

[3 marks]

Part of the solution curve is shown in the following diagram.



(b.i) Determine whether your answer to part (a) is an overestimate or an underestimate, justifying your answer. [2]

Markscheme

The estimate will be an overestimate ***A1***

because the tangents to the curve lie above the curve (the curve is concave down). ***R1***

[2 marks]

(b.ii) Justify why the use of Euler's method starting at $(1,\ 0)$ does not lead to an estimate of the negative value of y when $x = 2$. [1]

Markscheme

The gradient of the curve (at $(1,\ 0)$) is positive (so the method works in the direction of the upper part of the curve). ***R1***

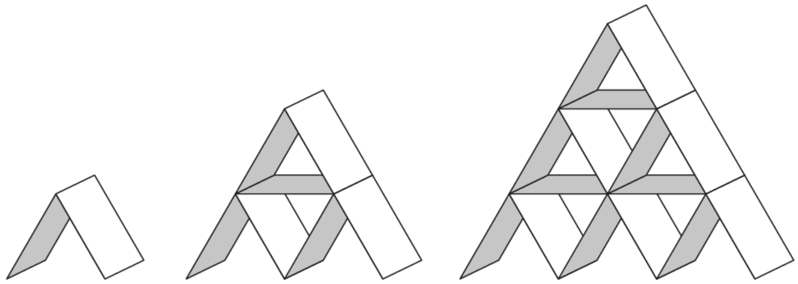
[1 mark]

10. [Maximum mark: 16] 25M.2.AHL.TZ1.10

Rectangular playing cards are stacked in the shape of a pyramid with n rows, where $n \geq 1$.

Some cards are placed horizontally and some cards are stacked at an angle of 60° to the horizontal.

The following diagrams represent pyramid stacks for $n = 1, n = 2$ and $n = 3$.



Let t_n represent the number of cards used to create a pyramid stack with n rows.

(a) Write down t_3. [1]

Markscheme

15 ***A1***

[1 mark]

(b) Find t_4. [2]

Markscheme

attempt to add 11 cards onto a stack with 3 rows **OR** attempt to consider all 4 rows ***(M1)***

valid diagram with 4 rows **OR** $t_4 = 15 + 11$ **OR** $t_4 = 2 + 5 + 8 + 11$

$= 26$ ***A1***

[2 marks]

(c) Show that $t_n = \frac{n(3n+1)}{2}$. [3]

Markscheme

METHOD 1

recognition that t_n is a sum of an arithmetic sequence ***(M1)***

$t_n = 2 + 5 + 8 + 11 + \ldots$

attempt to use formula for the sum of n terms of an arithmetic sequence ***M1***

$t_n = \frac{n}{2}(2(2) + 3(n-1))$ ***A1***

$t_n = \frac{n}{2}(3n+1)$ ***AG***

METHOD 2

attempt to split t_n into the total number of stacked and horizontal cards ***(M1)*** stacked $2+4+6+...=\frac{n}{2}(4+2(n-1))\left(=\frac{n}{2}(2n+2)\right)$ ***A1***

horizontal $0+1+2+...=\frac{n}{2}(0+1(n-1))\left(=\frac{n}{2}(n-1)\right)$ ***A1***

$t_n=\frac{n}{2}(4+2(n-1))+\frac{n}{2}(0+1(n-1))\left(=\frac{n}{2}(2n+2)+\frac{n}{2}(n-1)\right)$

$t_n=\frac{n}{2}(3n+1)$ ***AG***

METHOD 3

recognition that a stack with n rows is made up of complete triangles with the bottom row of horizontal cards removed and that the numbers of complete triangle cards form an arithmetic sequence ***(M1)***

$t_n=(3+6+9+12+...+3n)-n$ **OR** $t_n=3(1+2+3+4+...+n)-n$

attempt to use formula for the sum of n terms of an arithmetic sequence ***M1***

$t_n=\frac{n}{2}(2(3)+3(n-1))-n$ **OR** $t_n=3\times\frac{n}{2}(1+n)-n$ ***A1***

$t_n=\frac{n}{2}(3n+1)$ ***AG***

[3 marks]

There are 52 cards in a full pack of playing cards.

(d) A complete pyramid stack is created using playing cards taken from 14 full packs. Find the maximum number of rows in this stack. [3]

Markscheme

attempt to solve $\frac{n(3n+1)}{2}\leq 14(52)\ (=728)$ ***(M1)***

Note: Accept an attempt to solve an equation for ***(M1)***.

$21.8642...$ **OR** $n=21,\ t_n=672$ and $n=22,\ t_n=737$ ***(A1)***

max number of rows is 21 ***A1***

[3 marks]

(e) A complete pyramid stack is created using playing cards taken from full packs with no cards left over. Find the minimum number of rows in this stack. [2]

Markscheme

EITHER

attempt to solve by listing at least six values of t_n ***(M1)***

$2,\ 7,\ 15,\ 26,\ 40,\ 57\ldots$

OR

recognition that $\frac{\frac{1}{2}n(3n+1)}{52}$ must be an integer *(M1)*

$\frac{1}{2}n(3n+1) = 52k$ (where k is an integer)

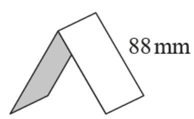
THEN

min number of rows is 13 ***A1***

Note: Award ***(M1)A0*** for an answer of 5 packs.
Award ***M0A0*** for any answer resulting from solving $\frac{1}{2}n(3n+1) = 52$.

[2 marks]

The long edge of each playing card measures $88\ \mathrm{mm}$ as illustrated in the following diagram.



(f) Find the minimum number of cards needed to create a complete pyramid stack with a vertical height of more than 2 metres. The thickness of the cards may be ignored. [5]

Markscheme

EITHER

attempt to use Pythagoras's Theorem or trigonometry to find the height of an equilateral triangle with sides $88\ \mathrm{mm}$ ***(M1)***

height $= \sqrt{88^2 - 44^2}$ **OR** $88\sin 60^\circ$ **OR** $88\cos 30^\circ$ **OR** $44\tan 60^\circ$ **OR** $\frac{44}{\tan 30^\circ}$ **OR** $44\sqrt{3} (= 76.2102\ldots)$ ***(A1)***

attempt to solve $44n\sqrt{3} > 2000$ **OR** their perpendicular height $\times n > 2000$ ***(M1)***

Note: Accept an attempt to solve an equation for ***(M1)***.

OR

attempt to use trigonometry to find the side of an equilateral triangle with height $2000\ \mathrm{mm}$ ***(M1)***

side $= \frac{2000}{\sin 60^\circ}$ **OR** $\frac{2000}{\cos 30^\circ}$ **OR** $\frac{4000}{\sqrt{3}} (= 2309.40\ldots)$ ***(A1)*** attempt to solve $88n > 2309.40\ldots$ **OR** $88n > \text{their side}$ ***(M1)***

Note: Accept an attempt to solve an equation for ***(M1)***.

THEN

$n > 26.2431\ldots$

so min number of rows is 27 ***(A1)***

$t_{27} = 1107$ ***A1***

[5 marks]

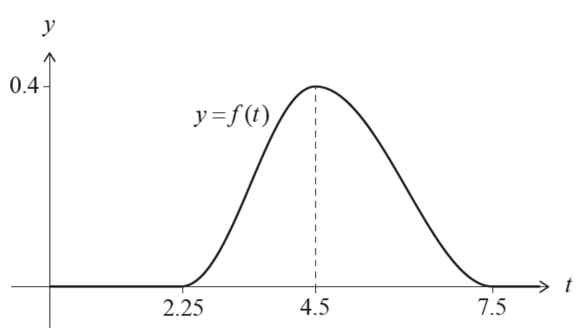
11. [Maximum mark: 19] 25M.2.AHL.TZ1.11

In a marathon race, the random variable T represents the time, in hours, taken for a runner to complete the race. No runner completes the race in less than 2.25 hours, and no runner completes it in more than 7.5 hours.

The probability density function for T is modelled by f, defined by

$$f(t) = \begin{cases} \frac{4}{21}\left(1 - \cos\left(\frac{4\pi}{9}(t - 2.25)\right)\right), & 2.25 \le t < 4.5 \\ \frac{4}{21}\left(1 + \cos\left(\frac{\pi}{3}(t - 4.5)\right)\right), & 4.5 \le t < 7.5 \\ 0, & \text{otherwise.} \end{cases}$$

The graph of f has a maximum point at $t = 4.5$ as shown in the following diagram:



(a.i) Find the value of $\int_{2.25}^{4.5} f(t)\ \mathrm{d}t$. [1]

Markscheme

$$\int_{2.25}^{4.5} \frac{4}{21}\left(1 - \cos\left(\frac{4\pi}{9}(t - 2.25)\right)\right)\ \mathrm{d}t$$

$= \frac{3}{7} (= 0.428571\ldots = 0.479)$ ***A1***

[1 mark]

(a.ii) Write down the mode of T. [1]

Markscheme

mode of $T = 4.5$ ***A1***

[1 mark]

(a.iii) Determine which is greater, the mode of T or the median of T, justifying your answer. [2]

Markscheme

the median is greater ***A1***

$\mathrm{P}(T < 4.5) < 0.5$ **OR** $\mathrm{P}(T < 4.5) > 0.5$ **OR** $\mathrm{P}(T < 4.5) < \mathrm{P}(T > 4.5)$ **OR** median $= 4.69$
R1

Note: Accept reference to areas rather than probabilities.

[2 marks]

The runners who finish the race in 3.5 hours or less are considered to be fast runners.

(b) Find the probability that a runner chosen at random is a fast runner. [2]

Markscheme

recognition of the need to integrate f ***(M1)***

$\int_{2.25}^{3.5} f(t)\, \mathrm{d}\, t$

$= 0.103749\ldots$

$= 0.104$ ***A1***

[2 marks]

(c) Find the probability that a fast runner chosen at random finishes the race in 3 hours or less. [3]

Markscheme

attempt to use formula for conditional probability in context ***(M1)***

$\frac{P(T \leq 3)}{P(\text{fast})}$ **OR** $\frac{P(T \leq 3)}{P(T \leq 3.5)}$ **OR** $\frac{P(\text{very fast})}{P(\text{fast})}$ (accept strict inequality signs)

$= \frac{0.0247152...}{0.103749...}$ ***(A1)***

$= 0.238220...$

$= 0.238$ ***A1***

[3 marks]

(d) Find the lower quartile of T. [3]

Markscheme

recognition that the lower quartile q is the value such that $\int_{2.25}^{q} f(t)\, \mathrm{d}t = 0.25$ ***(M1)***

$\int_{2.25}^{q} \frac{4}{21}\left(1 - \cos\left(\frac{4\pi}{9}(t - 2.25)\right)\right) \mathrm{d}t = 0.25$ ***(A1)***

Note: Condone the absence of $\mathrm{d}t$ for this ***A1***.

$\left[\frac{4}{21}\left(t - \frac{9}{4\pi}\sin\left(\frac{4\pi}{9}(t - 2.25)\right)\right)\right]_{2.25}^{q} = 0.25$

$q = 4.01290...$

$q = 4.01$ ***A1***

[3 marks]

Each runner's time is converted to a score which is calculated as $a - bt$, where t represents their time in hours, and $a,\ b > 0$.

Consider the random variable P which represents the score of a runner. It is given that $\mathrm{E}(P) = 100$ and the maximum possible score is 150.

(e) Use $\mathrm{E}(T) = 4.723$ to determine the value of a and the value of b, giving your answers to the nearest integer. [5]

Markscheme

attempt to find the expected value for a transformed linear variable ***(M1)***

$\mathrm{E}(P) = \mathrm{E}(a - bT) = a - b\mathrm{E}(T)$

$a - 4.723b = 100$ ***(A1)***

recognition that max score is achieved with fastest time $t = 2.25$ ***(M1)***

maximum score $a - 2.25b = 150$ ***(A1)***

$a = 195.491\ldots,\ b = 20.2183\ldots$

$a = 195,\ b = 20$ ***A1***

Note: these values must be given to the nearest integer for the ***A1*** to be awarded.

[5 marks]

(f) Given also that $\text{Var}\ (T) = 0.906$, find $\text{Var}\ (P)$. [2]

Markscheme

attempt to find variance for a transformed linear variable ***(M1)***

$\text{Var}\ (P) = \text{Var}\ (a - bT) = b^2\ \text{Var}\ (T)$

$0.906(20.2183\ldots^2) = 370.356\ldots$

$\text{Var}\ (P) = 370$ ***A1***

Note: accept any answer which rounds to any value between 362 and 370 inclusive based on use of less accurate values of b.

[2 marks]

12. [Maximum mark: 20] 25M.2.AHL.TZ1.12

Consider the family of functions f_n defined by $f_n(x) = \sum_{r=0}^{n} \left(-2x^2\right)^r$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Show that f_n is an even function for all values of n. [3]

Markscheme

attempt to replace x with $-x$ ***M1***

$f_n(-x) = \sum_{r=0}^{n} \left(-2(-x)^2\right)^r$ **OR** $-2(-x)^2 = -2x^2$ (seen anywhere) ***A1***

$f_n(-x) = f_n(x)$ ***A1***

so f_n is even for all values of n ***AG***

[3 marks]

(b.i) Show that $f_3(x) = 1 - 2x^2 + 4x^4 - 8x^6$. [1]

Markscheme

$f_3(x) = 1 - 2x^2 + \left(-2x^2\right)^2 + \left(-2x^2\right)^3$ ***A1***

$= 1 - 2x^2 + 4x^4 - 8x^6$ ***AG***

[1 mark]

(b.ii) Write down a similar expression for $f_4(x)$ in ascending powers of x. [1]

Markscheme

$= 1 - 2x^2 + 4x^4 - 8x^6 + 16x^8$ ***A1***

[1 mark]

Consider the function $f(x) = \lim_{n \to \infty} f_n(x)$ defined over the domain $-k < x < k$ where $k > 0$.

The largest possible value of k is K.

(c.i) Find the value of K, giving your answer in exact form. [3]

Markscheme

recognition of geometric series with common ratio $-2x^2$ ***(M1)***

converges for $\left|-2x^2\right| < 1 \left(\Rightarrow x^2 < \frac{1}{2}\right)$ ***(A1)***

$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \left(\Rightarrow -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}\right)$

largest $K = \frac{1}{\sqrt{2}} \left(= \frac{\sqrt{2}}{2}\right)$ ***A1***

[3 marks]

(c.ii) Express $f(x)$ as a rational function in the form $\frac{1}{a+bx^2}$, where a and b are constants to be determined. [2]

Markscheme

use of formula for S_∞ of a geometric series with first term 1, common ratio $-2x^2$ ***(M1)***

$f(x) = \frac{1}{1-(-2x^2)}$

$(f(x) =)\frac{1}{1+2x^2}$ ***A1***

$a = 1,\ b = 2$

[2 marks]

The function g is defined as $g(x) = f(x)$ for $0 \le x < K$.

(d.i) Justify that g^{-1} exists. [2]

Markscheme

g is a one-to-one function ***(A1)***

since g is a (strictly) decreasing function **OR** g has no points of zero gradient (turning points) ***R1***

[2 marks]

(d.ii) Find $g^{-1}(x)$, giving its domain. [4]

Markscheme

attempt to rearrange and swap x and y (at any stage) ***(M1)***

$x = \frac{1}{1+2y^2} \Rightarrow x + 2xy^2 = 1$

$y^2 = \frac{1-x}{2x}$ ***(A1)***

$y = \pm\sqrt{\frac{1-x}{2x}}$

$g^{-1}(x) = \sqrt{\frac{1-x}{2x}}$ ***A1***

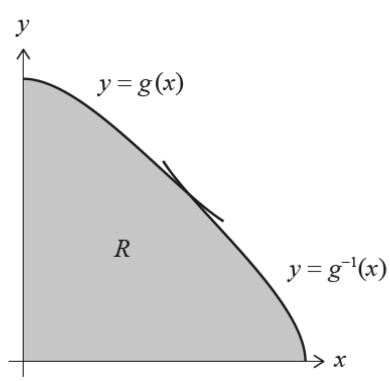
Note: Award ***A0*** if $g^{-1}(x)$ is missing.
Award ***FT A1*** marks only if f is of the correct form $f(x) = \frac{1}{a+bx^2}$

(domain) $\frac{1}{2} < x \leq 1$ ***A1***

Note: This ***A1*** can be awarded independently.

[4 marks]

The region R is completely enclosed by the curves $y = g(x),\ y = g^{-1}(x)$ and the x- and y-axes, as shown on the following diagram.



(e) Find the area of R. [4]

Markscheme

Note: Throughout part (e), do not award ***A1*** marks as ***FT*** from part (d)

METHOD 1

curves intersect at $x = 0.5897545107\ldots$ ***(A1)***

attempt to add the areas to the left and to the right of the point of intersection ***(M1)***

$$\int_0^{0.589\ldots} g(x)\, \mathrm{d}\, x + \int_{0.589\ldots}^{1} g^{-1}(x)\, \mathrm{d}\, x \left(= \int_0^{0.589\ldots} \frac{1}{1+2x^2}\, \mathrm{d}\, x + \int_{0.589\ldots}^{1} \sqrt{\frac{1-x}{2x}}\, \mathrm{d}\, x \right)$$

$= 0.491548\ldots + 0.143738\ldots$ ***(A1)***

Note: Award ***A1*** for one correct value seen, dependent on ***(M1)***

$= 0.635286\ldots$

Area of $R = 0.635$ ***A1***

METHOD 2

curves intersect at $x = 0.5897545107\ldots$

attempt to find the area between $y = g(x)$ and $y = x$ to the left of the point of intersection ***(M1)***

$$\int_0^{0.589\ldots} g(x)\, \mathrm{d}\, x - \int_0^{0.589\ldots} x\, \mathrm{d}\, x = \int_0^{0.589\ldots} \left(\frac{1}{1+2x^2} - x \right) \mathrm{d}\, x$$

$= 0.317643\ldots$ ***(A1)***

$2(0.3176432617\ldots) = 0.635286\ldots$

Area of $R = 0.635$ ***A1***

METHOD 3

curves intersect at $x = 0.5897545107\ldots$ ***A1***

attempt to find the area under $y = g^{-1}(x)$ and $y = x$ to the right of the point of intersection and the area of a square of side $x = 0.589\ldots$ ***(M1)***

$$0.589\ldots^2 + 2\int_{0.589\ldots}^{1} g^{-1}(x)\,\mathrm{d}\,x\left(= 0.589\ldots^2 + 2\int_{0.589\ldots}^{1}\sqrt{\frac{1-x}{2x}}\,\mathrm{d}\,x\right)$$

$= 0.347810\ldots + 2(0.143738\ldots)$ ***A1***

Note: Award ***A1*** for $0.143\ldots$ seen, dependent on ***(M1)***

$= 0.635286\ldots$

Area of $R = 0.635$ ***A1***

[4 marks]