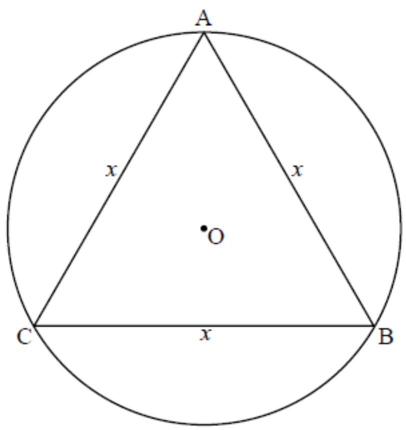
Paper 3 Examples [835 marks]

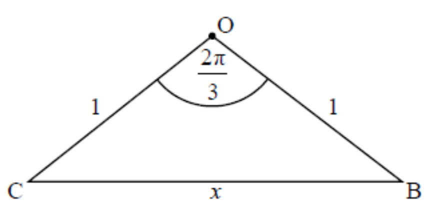
1. [Maximum mark: 30] SPM.3.AHL.TZ0.1

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

(a) Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram.



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units. [3]

Markscheme

METHOD 1

consider right-angled triangle OCX where CX $= \frac{x}{2}$

$\sin\frac{\pi}{3} = \frac{\frac{x}{2}}{1}$ ***M1A1***

$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2\,(1)\,(1)\cos\frac{2\pi}{3}$ ***M1A1***

$x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

Note: Accept use of sine rule.

[3 marks]

(b) Consider a square of side length, x units, inscribed in a circle of radius 1 unit. By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. [3]

Markscheme

$\sin\frac{\pi}{4} = \frac{1}{x}$ where x = side of square ***M1***

$x = \sqrt{2}$ ***A1***

$P_i = 4\sqrt{2}$ ***A1***

[3 marks]

(c) Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. [2]

Markscheme

6 equilateral triangles ⇒x = 1 ***A1***

$P_i = 6$ ***A1***

[2 marks]

Let $P_i\left(n\right)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

(d) Show that $P_i\left(n\right) = 2n\sin\left(\frac{\pi}{n}\right)$. [3]

Markscheme

in right-angled triangle $\sin\left(\frac{\pi}{n}\right) = \frac{\frac{x}{2}}{1}$ ***M1***

$\Rightarrow x = 2\sin\left(\frac{\pi}{n}\right)$ ***A1***

$P_i = n \times x$

$P_i = n \times 2\sin\left(\frac{\pi}{n}\right)$ ***M1***

$P_i = 2n\sin\left(\frac{\pi}{n}\right)$ ***AG***

[3 marks]

(e) Use an appropriate Maclaurin series expansion to find $\lim\limits_{n\to\infty} P_i\left(n\right)$ and interpret this result geometrically. [5]

Markscheme

consider $\lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ ***M1***

$2n\sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \ldots\right)$ ***(A1)***

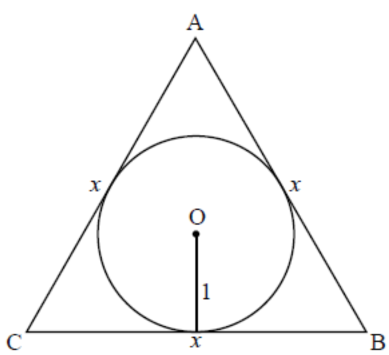
$= 2\left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \ldots\right)$ ***A1***

$\Rightarrow \lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\pi$ ***A1***

as $n \to \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$ ***R1***

[5 marks]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

(f) Show that $P_c(n) = 2n\tan\left(\frac{\pi}{n}\right)$. [4]

Markscheme

consider an n-sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre ***M1A1***

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2\tan\left(\frac{\pi}{n}\right)$ ***M1A1***

Perimeter $P_c = 2n\tan\left(\frac{\pi}{n}\right)$ ***AG***

[4 marks]

(g) By writing $P_c(n)$ in the form $\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}$, find $\lim_{n\to\infty} P_c(n)$. [5]

Markscheme

consider $\lim_{n\to\infty} 2n\tan\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right) = \frac{0}{0}$ ***R1***

attempt to use L'Hopital's rule ***M1***

$= \lim_{n\to\infty}\left(\frac{-\frac{2\pi}{n^2}\sec^2\left(\frac{\pi}{n}\right)}{-\frac{1}{n^2}}\right)$ ***A1A1***

$= 2\pi$ ***A1***

[5 marks]

(h) Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. [2]

Markscheme

$P_i < 2\pi < P_c$

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$ ***M1***

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$ ***A1***

[2 marks]

(i) The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π.

Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π. [3]

Markscheme

attempt to find the lower bound and upper bound approximations within 0.005 of π ***(M1)***

$n = 46$ ***A2***

[3 marks]

2. [Maximum mark: 25] SPM.3.AHL.TZ0.2

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

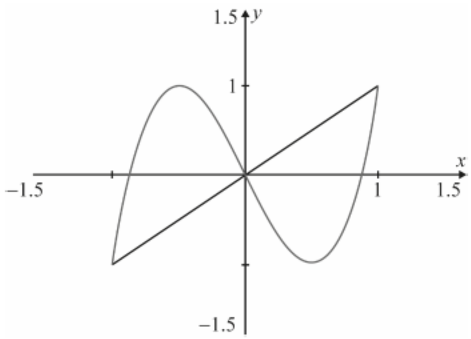
Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$. [2]

Markscheme

correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(b.i) local maximum points; [3]

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

[3 marks]

(b.ii) local minimum points; [1]

Markscheme

$\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

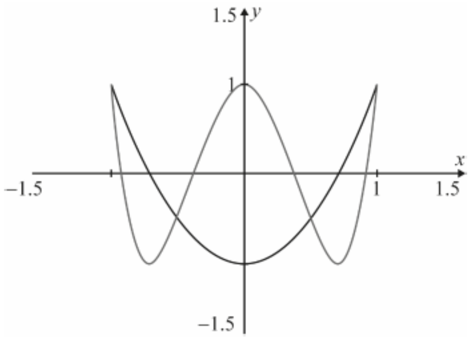
[1 mark]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$. [2]

Markscheme

correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of $n$describing, in terms of n, the number of

(d.i) local maximum points; [3]

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 2$, 0 local maximum point and 1 local minimum point

$n = 4$, 1 local maximum points and 2 local minimum points

$n = 6$, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

[3 marks]

(d.ii) local minimum points. [1]

Markscheme

$\frac{n}{2}$ local minimum points ***A1***

[1 mark]

(e) Solve the equation $f_n{}'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

Markscheme

$f_n(x) = \cos(n \arccos(x))$

$f_n{}'(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f_n{}'(x) = 0 \Rightarrow n \sin(n \arccos(x)) = 0$ ***M1***

$n \arccos(x) = k\pi \quad (k \in \mathbb{Z}^+)$ ***A1*** leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and $0 < k < n$) ***AG***

[4 marks]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Markscheme

$f_2(x) = \cos(2\arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

(g) Use an appropriate trigonometric identity to show that
$f_{n+1}(x) = \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$. [2]

Markscheme

$f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A + B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h.i) Hence show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x), n \in \mathbb{Z}^+$. [3]

Markscheme

$f_{n-1}(x) = \cos((n-1)\arccos x)$ ***A1***

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ ***A1***

$= 2xf_n(x)$ ***AG***

[3 marks]

(h.ii) Hence express $f_3(x)$ as a cubic polynomial. [2]

Markscheme

$f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x\left(2x^2 - 1\right) - x$

$= 4x^3 - 3x$ ***A1***

[2 marks]

3. [Maximum mark: 30] EXN.3.AHL.TZ0.2

A **Gaussian integer** is a complex number, z, such that $z = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\beta = 1 - 2\mathrm{i}$, such that $\gamma = \alpha\beta$ for some Gaussian integer γ.

(a) Find γ. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(3 + 4\mathrm{i})(1 - 2\mathrm{i}) = 11 - 2\mathrm{i}$ ***(M1)A1***

[2 marks]

Now consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\gamma = 11 + 2\mathrm{i}$.

(b) Determine whether $\frac{\gamma}{\alpha}$ is a Gaussian integer. [3]

Markscheme

$\frac{\gamma}{\alpha} = \frac{41}{25} - \frac{38}{25}\mathrm{i}$ ***(M1)A1***

(Since $\mathrm{Re}\frac{\gamma}{\alpha}\left(= \frac{41}{25}\right)$ and/or $\mathrm{Im}\frac{\gamma}{\alpha}\left(= -\frac{38}{25}\right)$ are not integers)

$\frac{\gamma}{\alpha}$ is not a Gaussian integer ***R1***

Note: Award ***R1*** for correct conclusion from their answer.

[3 marks]

The norm of a complex number z, denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3\mathrm{i}$ then $N(2 + 3\mathrm{i}) = 2^2 + 3^2 = 13$.

(c) On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3. [2]

Markscheme

$\pm 1,\ \pm \mathrm{i},\ 0$ plotted and labelled ***A1***

$1 \pm \mathrm{i},\ -1 \pm \mathrm{i}$ plotted and labelled ***A1***

Note: Award ***A1A0*** if extra points to the above are plotted and labelled.

[2 marks]

(d) Given that $\alpha = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$. [1]

Markscheme

$|z| = \sqrt{a^2 + b^2}$ (and as $N(z) = |z|^2$) ***A1***

then $N(\alpha) = a^2 + b^2$ ***AG***

[1 mark]

A **Gaussian prime** is a Gaussian integer, z, that **cannot** be expressed in the form $z = \alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers with $N(\alpha),\ N(\beta) > 1$.

(e) By expressing the positive integer $n = c^2 + d^2$ as a product of two Gaussian integers each of norm $c^2 + d^2$, show that n is not a Gaussian prime. [3]

Markscheme

$c^2 + d^2 = (c + d\mathrm{i})(c - d\mathrm{i})$ ***A1***

and $N(c + d\mathrm{i}) = N(c - d\mathrm{i}) = c^2 + d^2$ ***R1***

$N(c + d\mathrm{i}),\ N(c - d\mathrm{i}) > 1$ (since $c,\ d$ are positive) ***R1***

so $c^2 + d^2$ is not a Gaussian prime, by definition ***AG***

[3 marks]

The positive integer 2 is a prime number, however it is not a Gaussian prime.

(f) Verify that 2 is not a Gaussian prime. [2]

Markscheme

$2\left(= 1^2 + 1^2\right) = (1 + \mathrm{i})(1 - \mathrm{i})$ ***(A1)***

$N(1 + \mathrm{i}) = N(1 - \mathrm{i}) = 2$ ***A1*** so 2 is not a Gaussian prime ***AG***

[2 marks]

(g) Write down another prime number of the form $c^2 + d^2$ that is not a Gaussian prime and express it as a product of two Gaussian integers. [2]

Markscheme

For example, $5\left(= 1^2 + 2^2\right) = (1 + 2\text{i})(1 - 2\text{i})$ ***(M1)A1***

[2 marks]

Let $\alpha,\ \beta$ be Gaussian integers.

(h) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. [6]

Markscheme

METHOD 1

Let $\alpha = m + n\text{i}$ and $\beta = p + q\text{i}$

LHS:

$\alpha\beta = (mp - nq) + (mq + np)\text{i}$ ***M1***

$N(\alpha\beta) = (mp - nq)^2 + (mq + np)^2$ ***A1***

$(mp)^2 - 2mnpq + (nq)^2 + (mq)^2 + 2mnpq + (np)^2$ ***A1***

$(mp)^2 + (nq)^2 + (mq)^2 + (np)^2$ ***A1***

RHS:

$N(\alpha)N(\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$(mp)^2 + (mq)^2 + (np)^2 + (nq)^2$ ***A1***

LHS = RHS and so $N(\alpha\beta) = N(\alpha)N(\beta)$ ***AG***

METHOD 2

Let $\alpha = m + n\text{i}$ and $\beta = p + q\text{i}$

LHS

$N(\alpha\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$= (m + n\text{i})(m - n\text{i})(p + q\text{i})(p - q\text{i})$ ***A1***

$= (m + n\text{i})(p + q\text{i})(m - n\text{i})(p - q\text{i})$

$= ((mp - nq) + (mq + np)\mathrm{i})((mp - nq) - (mq + np)\mathrm{i})$ ***M1A1***

$= (mp - nq)^2 + (mq + np)^2$ ***A1***

$N = ((mp - nq) + (mq + np)\mathrm{i})$ ***A1***

$= N(\alpha)N(\beta)$ (= RHS) ***AG***

[6 marks]

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

(i) Hence show that $1 + 4\mathrm{i}$ is a Gaussian prime. [3]

Markscheme

$N(1 + 4\mathrm{i}) = 17$ which is a prime (in $\mathbb{Z}$) ***R1***

if $1 + 4\mathrm{i} = \alpha\beta$ then $17 = N(\alpha\beta) = N(\alpha)N(\beta)$ ***R1***

we cannot have $N(\alpha),\ N(\beta) > 1$ ***R1***

Note: Award ***R1*** for stating that $1 + 4\mathrm{i}$ is not the product of Gaussian integers of smaller norm because no such norms divide 17

so $1 + 4\mathrm{i}$ is a Gaussian prime ***AG***

[3 marks]

(j) Use proof by contradiction to prove that a prime number, p, that is not of the form $a^2 + b^2$ is a Gaussian prime. [6]

Markscheme

Assume p is not a Gaussian prime

$\Rightarrow p = \alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers and $N(\alpha),\ N(\beta) > 1$ ***M1***

$\Rightarrow N(p) = N(\alpha)N(\beta)$ ***M1***

$p^2 = N(\alpha)N(\beta)$ ***A1***

It cannot be $N(\alpha) = 1,\ N(\beta) = p^2$ from definition of Gaussian prime ***R1***

hence $N(\alpha) = p,\ N(\beta) = p$ ***R1***

If $\alpha = a + b\mathrm{i}$ then $N(\alpha) = a^2 + b^2 = p$ which is a contradiction ***R1***

hence a prime number, p, that is not of the form $a^2 + b^2$ is a Gaussian prime ***AG***

[6 marks]

4. [Maximum mark: 27] EXM.3.AHL.TZ0.1

This question will investigate power series, as an extension to the Binomial Theorem for negative and fractional indices.

A power series in x is defined as a function of the form $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ where the $a_i \in \mathbb{R}$.

It can be considered as an infinite polynomial.

(a) Expand $(1+x)^5$ using the Binomial Theorem. [2]

Markscheme

$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ ***M1A1***

[2 marks]

This is an example of a power series, but is only a finite power series, since only a finite number of the a_i are non-zero.

(b) Consider the power series $1 - x + x^2 - x^3 + x^4 - \ldots$

By considering the ratio of consecutive terms, explain why this series is equal to $(1+x)^{-1}$ and state the values of x for which this equality is true. [4]

Markscheme

It is an infinite GP with $a = 1,\ r = -x$ ***R1A1***

$S_\infty = \frac{1}{1-(-x)} = \frac{1}{1+x} = (1+x)^{-1}$ ***M1A1AG***

[4 marks]

(c) Differentiate the equation obtained part (b) and hence, find the first four terms in a power series for $(1+x)^{-2}$. [2]

Markscheme

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$

$-1(1+x)^{-2} = -1 + 2x - 3x^2 + 4x^3 - \ldots$ ***A1***

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$ ***A1***

[2 marks]

(d) Repeat this process to find the first four terms in a power series for $(1+x)^{-3}$. [2]

Markscheme

$-2(1+x)^{-3} = -2 + 6x - 12x^2 + 20x^3 \ldots$ ***A1***

$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 \ldots$ ***A1***

[2 marks]

(e) Hence, by recognising the pattern, deduce the first four terms in a power series for $(1+x)^{-n}, n \in \mathbb{Z}^+$. [3]

Markscheme

$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 \ldots$ ***A1A1A1***

[3 marks]

We will now attempt to generalise further.

Suppose $(1+x)^q,\ q \in \mathbb{Q}$ can be written as the power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$.

(f) By substituting $x = 0$, find the value of a_0. [1]

Markscheme

$1^q = a_0 \Rightarrow a_0 = 1$ ***A1***

[1 mark]

(g) By differentiating both sides of the expression and then substituting $x = 0$, find the value of a_1. [2]

Markscheme

$q(1+x)^{q-1} = a_1 + 2a_2x + 3a_3x^2 + \ldots$ ***A1***

$a_1 = q$ ***A1***

[2 marks]

(h) Repeat this procedure to find a_2 and a_3. [4]

Markscheme

$q\,(q-1)(1+x)^{q-2} = 1 \times 2a_2 + 2 \times 3a_3x + \ldots$ ***A1***

$a_2 = \frac{q(q-1)}{2!}$ ***A1***

$q\,(q-1)\,(q-2)(1+x)^{q-3} = 1 \times 2 \times 3a_3 + \ldots$ ***A1***

$a_3 = \frac{q(q-1)(q-2)}{3!}$ ***A1***

[4 marks]

(i) Hence, write down the first four terms in what is called the Extended Binomial Theorem for $(1+x)^q,\ q \in \mathbb{Q}$. [1]

Markscheme

$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 \ldots$ ***A1***

[1 mark]

(j) Write down the power series for $\frac{1}{1+x^2}$. [2]

Markscheme

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ ***M1A1***

[2 marks]

(k) Hence, using integration, find the power series for $\arctan x$, giving the first four non-zero terms. [4]

Markscheme

$\arctan x + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***M1A1***

Putting $x = 0 \Rightarrow c = 0$ ***R1***

So $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***A1***

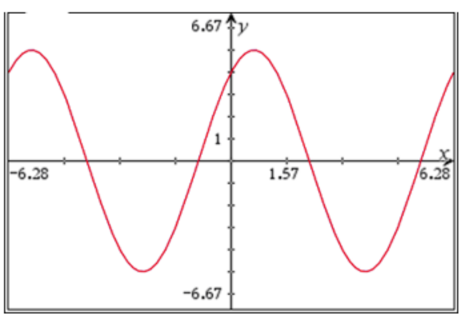
[4 marks]

5. [Maximum mark: 26] EXM.3.AHL.TZ0.5

This question investigates the sum of sine and cosine functions

(a.i) Sketch the graph $y = 3\sin x + 4\cos x$, for $-2\pi \leqslant x \leqslant 2\pi$ [1]

Markscheme



A1

[1 mark]

(a.ii) Write down the amplitude of this graph [1]

Markscheme

5 ***A1***

[1 mark]

(a.iii) Write down the period of this graph [1]

Markscheme

2π ***A1***

[1 mark]

The expression $3\sin x + 4\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

(b.i) Use your answers from part (a) to write down the value of A, B and D. [1]

Markscheme

$A = 5, B = 1, D = 0$ ***A1***

[1 mark]

(b.ii) Find the value of C. [2]

Markscheme

maximum at $x = 0.644$ ***M1***

So $C = -0.644$ ***A1***

[2 marks]

(c.i) Find $\arctan\frac{3}{4}$, giving the answer to 3 significant figures. [1]

Markscheme

0.644 ***A1***

[1 mark]

(c.ii) Comment on your answer to part (c)(i). [1]

Markscheme

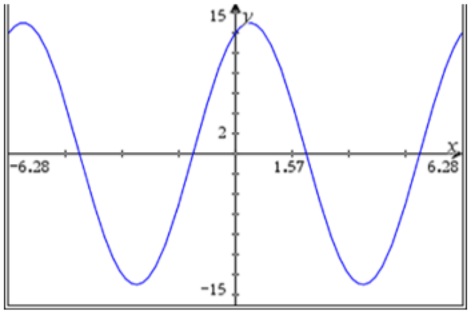
it appears that $C = -\arctan\frac{3}{4}$ ***A1***

[1 mark]

The expression $5\sin x + 12\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

(d) By considering the graph of $y = 5\sin x + 12\cos x$, find the value of A, B, C and D. [5]

Markscheme



M1

$A = 13$ ***A1***

$B = 1$ and $D = 0$ ***A1***

maximum at $x = 0.395$ ***M1***

So C = −0.395 $\left(= -\arctan\frac{5}{12}\right)$ ***A1***

[5 marks]

In general, the expression $a \sin x + b \cos x$ can be written in the form $A \cos(Bx + C) + D$, where $a,\ b,\ A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

Conjecture an expression, in terms of a and b, for

(e.i) A. [1]

Markscheme

$A = \sqrt{a^2 + b^2}$ ***A1***

[1 mark]

(e.ii) B. [1]

Markscheme

$B = 1$ ***A1***

[1 mark]

(e.iii) C. [1]

Markscheme

$C = -\arctan\frac{a}{b}$ ***A1***

[1 mark]

(e.iv) D. [1]

Markscheme

$D = 0$ ***A1***

[1 mark]

The expression $a\sin x + b\cos x$ can also be written in the form $\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$.

Let $\frac{a}{\sqrt{a^2+b^2}} = \sin\theta$

(f.i) Show that $\frac{b}{\sqrt{a^2+b^2}} = \cos\theta$. [2]

Markscheme

EITHER

use of a right triangle and Pythgoras' to show the missing side length is b ***M1A1***

OR

Use of $\sin^2\theta + \cos^2\theta = 1$, leading to the required result ***M1A1***

[2 marks]

(f.ii) Show that $\frac{a}{b} = \tan\theta$. [1]

Markscheme

EITHER

use of a right triangle, leading to the required result. ***M1***

OR

Use of $\tan\theta = \frac{\sin\theta}{\cos\theta}$, leading to the required result. ***M1***

[1 mark]

(g) Hence prove your conjectures in part (e). [6]

Markscheme

$a\sin x + b\cos x = \sqrt{a^2+b^2}\,(\sin\theta\sin x + \cos\theta\cos x)$ ***M1***

$a\sin x + b\cos x = \sqrt{a^2+b^2}\,(\cos(x-\theta))$ ***M1A1***

So $A = \sqrt{a^2+b^2}, B = 1$ and $D = 0$ ***A1***

And $C = -\theta$ ***M1***

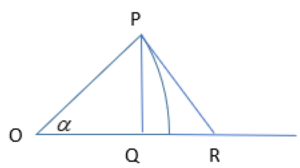
So $C = -\arctan\frac{a}{b}$ ***A1***

[6 marks]

6. [Maximum mark: 28] EXM.3.AHL.TZ0.2

This question will explore connections between complex numbers and regular polygons.

The diagram below shows a sector of a circle of radius 1, with the angle subtended at the centre O being $\alpha,\ 0 < \alpha < \frac{\pi}{2}$. A perpendicular is drawn from point P to intersect the x-axis at Q. The tangent to the circle at P intersects the x-axis at R.



(a) By considering the area of two triangles and the area of the sector show that $\cos\alpha\sin\alpha < \alpha < \frac{\sin\alpha}{\cos\alpha}$. [5]

Markscheme

Area triangle $OPQ = \frac{1}{2}\cos\alpha\sin\alpha$ ***A1***

Area sector $= \frac{1}{2}1^2\alpha$ ***A1***

Area triangle $OPR = \frac{1}{2}1\tan\alpha$ ***A1***

So looking at the diagram $\frac{1}{2}\cos\alpha\sin\alpha < \frac{1}{2}\alpha < \frac{1}{2}\frac{\sin\alpha}{\cos\alpha}$ ***M1***

$\Rightarrow \cos\alpha\sin\alpha < \alpha < \frac{\sin\alpha}{\cos\alpha}$ ***AG***

[5 marks]

(b) Hence show that $\lim\limits_{\alpha\to 0}\frac{\alpha}{\sin\alpha} = 1$. [2]

Markscheme

Hence $\cos\alpha < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}$ and as $\alpha \to 0,\ \cos\alpha \to 1$ we have ***M1R1***

$\lim\limits_{\alpha\to 0}\frac{\alpha}{\sin\alpha} = 1$ ***AG***

[2 marks]

(c) Let $z^n = 1,\ z \in \mathbb{C},\ n \in \mathbb{N},\ n \geqslant 5$. Working in modulus/argument form find the n solutions to this equation. [8]

Markscheme

$(r\,cis\,\theta)^n = 1\,cis\,0 \Rightarrow r^n cis\,n\,\theta = 1\,cis\,\theta$ ***M1A1M1A1***

$r^n = 1 \Rightarrow r = 1 \quad n\theta = 0 + 2\pi k,\ k \in \mathbb{Z}$ ***A1A1***

$\theta = \frac{2\pi k}{n},\ 0 \leqslant k \leqslant n-1$ ***A1***

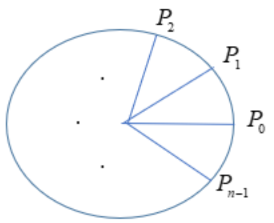
$z = cis\frac{2\pi k}{n},\ 0 \leqslant k \leqslant n-1$ ***A1***

[8 marks]

(d) Represent these n solutions on an Argand diagram. Let their positions be denoted by $P_0,\ P_1,\ P_2,\ \ldots P_{n-1}$ placed in order in an anticlockwise direction round the circle, starting on the positive x-axis. Show the positions of $P_0,\ P_1,\ P_2$ and P_{n-1}.

[1]

Markscheme



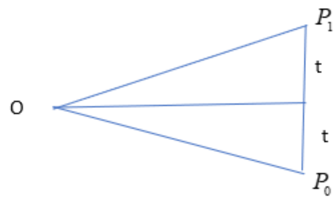
A1

[1 mark]

(e) Show that the length of the line segment P_0P_1 is $2\sin\frac{\pi}{n}$. [4]

Markscheme

Bisecting the triangle OP_0P_1 to form two right angle triangles *M1*



Length of $P_0P_1 = 2t$ where $t = \sin\left(\frac{\frac{2\pi}{n}}{2}\right)$ *M1A1A1*

So length is $2\sin\frac{\pi}{n}$ *AG*

[4 marks]

(f) Hence, write down the total length of the perimeter of the regular n sided polygon $P_0P_1P_2\ldots P_{n-1}P_0$. [1]

Markscheme

Length of perimeter is $2n\sin\frac{\pi}{n}$ *A1*

[1 mark]

(g) Using part (b) find the limit of this perimeter as $n\to\infty$. [2]

Markscheme

$2n\sin\frac{\pi}{n} = 2\pi\frac{n}{\pi}\sin\frac{\pi}{n} \to 2\pi$ as $n\to\infty$ *M1A1*

[2 marks]

(h) Find the total area of this n sided polygon. [3]

Markscheme

Area of $OP_0P_1 = \frac{1}{2}1 \times 1 \sin\frac{2\pi}{n}$ so total area is $\frac{n}{2}\sin\frac{2\pi}{n}$. ***M1A1A1***

[3 marks]

(i) Using part (b) find the limit of this area as $n \to \infty$. [2]

Markscheme

$\frac{n}{2}\sin\frac{2\pi}{n} = \pi\frac{n}{2\pi}\sin\frac{2\pi}{n} \to \pi$ as $n \to \infty$ ***M1A1***

[2 marks]

7. [Maximum mark: 35] EXM.3.AHL.TZ0.4

This question investigates some applications of differential equations to modeling population growth.

One model for population growth is to assume that the rate of change of the population is proportional to the population, i.e. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where $k \in \mathbb{R}$, t is the time (in years) and P is the population

(a) Show that the general solution of this differential equation is $P = A\mathrm{e}^{kt}$, where $A \in \mathbb{R}$. [5]

Markscheme

$\int \frac{1}{P}\mathrm{d}P = \int k\mathrm{d}t$ ***M1A1***

$\ln P = kt + c$ ***A1A1***

$P = e^{kt+c}$ ***A1***

$P = Ae^{kt}$, where $A = e^c$ ***AG***

[5 marks]

The initial population is 1000.

Given that $k = 0.003$, use your answer from part (a) to find

(b.i) the population after 10 years [2]

Markscheme

when $t = 0,\ P = 1000$

$\Rightarrow A = 1000$ ***A1***

$P(10) = 1000e^{0.003(10)} = 1030$ ***A1***

[2 marks]

(b.ii) the number of years it will take for the population to triple. [2]

Markscheme

$3000 = 1000e^{0.003t}$ ***M1***

$t = \frac{\ln 3}{0.003} = 366$ years ***A1***

[2 marks]

(b.iii) $\lim_{t \to \infty} P$ [1]

Markscheme

$\lim_{t \to \infty} P = \infty$ ***A1***

[1 mark]

Consider now the situation when k is not a constant, but a function of time.

Given that $k = 0.003 + 0.002t$, find

(c.i) the solution of the differential equation, giving your answer in the form $P = f(t)$. [5]

Markscheme

$\int \frac{1}{P} \mathrm{d}P = \int (0.003 + 0.002t)\mathrm{d}t$ ***M1***

$\ln P = 0.003t + 0.001t^2 + c$ ***A1A1***

$P = e^{0.003t + 0.001t^2 + c}$ ***A1***

when $t = 0,\ P = 1000$

$\Rightarrow e^c = 1000$ ***M1***

$P = 1000e^{0.003t + 0.001t^2}$

[5 marks]

(c.ii) the number of years it will take for the population to triple. [4]

Markscheme

$3000 = 1000e^{0.003t + 0.001t^2}$ ***M1***

$\ln 3 = 0.003t + 0.001t^2$ ***A1***

Use of quadratic formula or GDC graph or GDC polysmlt ***M1***

$t = 31.7$ years ***A1***

[4 marks]

Another model for population growth assumes

- there is a maximum value for the population, L.
- that k is not a constant, but is proportional to $\left(1 - \frac{P}{L}\right)$.

(d) Show that $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P(L-P)$, where $m \in \mathbb{R}$. [2]

Markscheme

$k = m\left(1 - \frac{P}{L}\right)$, where m is the constant of proportionality ***A1***

So $\frac{\mathrm{d}P}{\mathrm{d}t} = m\left(1 - \frac{P}{L}\right)P$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P(L-P)$ ***AG***

[2 marks]

(e) Solve the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P(L-P)$, giving your answer in the form $P = g(t)$. [10]

Markscheme

$\int \frac{1}{P(L-P)}\mathrm{d}P = \int \frac{m}{L}\mathrm{d}t$ ***M1***

$\frac{1}{P(L-P)} = \frac{A}{P} + \frac{B}{L-P}$ ***M1***

$1 \equiv A(L-P) + BP$ ***A1***

$A = \frac{1}{L},\ B = \frac{1}{L}$ ***A1***

$\frac{1}{L}\int\left(\frac{1}{P} + \frac{1}{L-P}\right)\mathrm{d}P = \int \frac{m}{L}\mathrm{d}t$

$\frac{1}{L}(\ln P - \ln(L-P)) = \frac{m}{L}t + c$ ***A1A1***

$\ln\left(\frac{P}{L-P}\right) = mt + d$, where $d = cL$ ***M1***

$\frac{P}{L-P} = Ce^{mt}$, where $C = e^{d}$ ***A1***

$P\left(1 + Ce^{mt}\right) = CLe^{mt}$ ***M1***

$P = \frac{CLe^{mt}}{(1+Ce^{mt})}\ \left(= \frac{L}{(De^{-mt}+1)},\ \text{where } D = \frac{1}{C}\right)$ ***A1***

[10 marks]

(f) Given that the initial population is 1000, $L = 10000$ and $m = 0.003$, find the number of years it will take for the population to triple. [4]

Markscheme

$1000 = \frac{10000}{D+1}$ ***M1***

$D = 9$ ***A1***

$3000 = \frac{10000}{9e^{-0.003t}+1}$ ***M1***

$t = 450$ years ***A1***

[4 marks]

8. [Maximum mark: 29] EXM.3.AHL.TZ0.3

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int\limits_0^{\frac{\pi}{2}} \sin^n x \, dx, \; n \in \mathbb{N}$.

(a) Find the exact values of I_0, I_1 and I_2. [6]

Markscheme

$I_0 = \int\limits_0^{\frac{\pi}{2}} 1 \, dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$ ***M1A1***

$I_1 = \int\limits_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$ ***M1A1***

$I_2 = \int\limits_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int\limits_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} \, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ ***M1A1***

[6 marks]

(b.i) Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}, \; n \geqslant 2$. [5]

Markscheme

$u = \sin^{n-1} x$ $\qquad$ $v = -\cos x$

$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$ $\qquad$ $\frac{dv}{dx} = \sin x$

$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int\limits_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos^2 x \, dx$ ***M1A1A1***

$= 0 + \int\limits_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \left(1 - \sin^2 x\right) dx = (n-1)\left(I_{n-2} - I_n\right)$ ***M1A1***

$\Rightarrow nI_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{(n-1)}{n} I_{n-2}$ ***AG***

[6 marks]

(b.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

Markscheme

need $n \geqslant 2$ so that $\sin^{n-1} \frac{\pi}{2} = 0$ in $\left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}}$ ***R1***

[1 mark]

(c) Hence, find the exact values of I_3 and I_4. [2]

Markscheme

$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$ $\qquad$ $I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}$ ***A1A1***

[2 marks]

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx, \; n \in \mathbb{N}$.

(d) Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. [4]

Markscheme

$x = \frac{\pi}{2} - u \Rightarrow \frac{dx}{du} = -1$ ***A1***

$J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \int_{\frac{\pi}{2}}^{0} -\cos^n \left(\frac{\pi}{2} - u\right) du = -\int_{\frac{\pi}{2}}^{0} \sin^n u \, du = \int_0^{\frac{\pi}{2}} \sin^n u \, du = I_n$ ***M1A1A1AG***

[4 marks]

(e) Hence, find the exact values of J_5 and J_6 [2]

Markscheme

$J_5 = I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$ $\quad J_6 = I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3\pi}{16} = \frac{5\pi}{32}$ ***A1A1***

[2 marks]

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \; n \in \mathbb{N}$.

(f) Find the exact values of T_0 and T_1. [3]

Markscheme

$T_0 = \int_0^{\frac{\pi}{4}} 1 \, dx = [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$ ***A1***

$T_1 = \int_0^{\frac{\pi}{4}} \tan dx = [-\ln|\cos x|]_0^{\frac{\pi}{4}} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$ ***M1A1***

[3 marks]

(g.i) Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2}, \; n \geqslant 2$. [3]

Markscheme

$T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\sec^2 x - 1\right) dx$ ***M1***

$\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - T_{n-2} = \frac{1}{n-1} - T_{n-2}$ ***A1A1AG***

[3 marks]

(g.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

Markscheme

need $n \geqslant 2$ so that the powers of tan in $\int_0^{\frac{\pi}{4}} \tan^{n-2}x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2}x \, dx$ are not negative ***R1***

[1 mark]

(h) Hence, find the exact values of T_2 and T_3. [2]

Markscheme

$T_2 = 1 - T_0 = 1 - \frac{\pi}{4}$ ***A1***

$T_3 = \frac{1}{2} - T_1 = \frac{1}{2} - \ln\sqrt{2}$ ***A1***

[2 marks]

9. [Maximum mark: 26] 25N.3.AHL.TZ1.1

This question asks you to investigate lines normal to curves of the form $y = \frac{k^2}{x}$.

The curve H has equation $y = \frac{1}{x}$ where $x \in \mathbb{R}, x \neq 0$.

(a) A line N is normal to H at $x = t$.

(a.i) Show that the gradient of N is t^2. [2]

Markscheme

METHOD 1

attempts to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}$ **AND** $m_N = \frac{-1}{-\frac{1}{t^2}}$ (or equivalent) ***A1***

Note: Accept $m_1 m_2 = -1$ (or equivalent) seen anywhere.
Award ***A1*** for $m_N = x^2$ and as $x = t$, then $m_N = t^2$.

so gradient of N is t^2. ***AG***

METHOD 2

uses $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{t^2}}{1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}$ **AND** $m_N = \frac{-1}{-\frac{1}{t^2}}$ (or equivalent) ***A1***

Note: Accept $m_1 m_2 = -1$ (or equivalent) seen anywhere.
Award ***A1*** for $m_N = x^2$ and as $x = t$, then $m_N = t^2$.

so gradient of N is t^2 ***AG***

[2 marks]

(a.ii) Hence, show that the equation of N is $y = t^2 x + \frac{1}{t} - t^3$. [1]

Markscheme

EITHER

$y - \frac{1}{t} = t^2(x - t)$ **OR** $y = t^2(x - t) + \frac{1}{t}$ ***A1***

OR

$\frac{1}{t} = t^2(t) + c \Rightarrow c = \frac{1}{t} - t^3$ ***A1***

OR

$t^2 x - y - \left(t^2(t) - \frac{1}{t}\right) = 0$ **OR** $t^2 x - y = t^2(t) - \frac{1}{t}$ ***A1***

THEN

leading to $y = t^2 x + \frac{1}{t} - t^3$ ***AG***

[1 mark]

(b) The equation for N given in part (a)(ii) is of the form $y = mx + c$.

(b.i) Show that either $c = \frac{1}{\sqrt{m}}\left(1 - m^2\right)$ or $c = \frac{1}{\sqrt{m}}\left(m^2 - 1\right)$. [4]

Markscheme

$m = t^2$ $\left(m^2 = t^4\right)$ **OR** $c = \frac{1}{t} - t^3$ **OR** $c = \frac{1}{t}\left(1 - t^4\right)$ ***(A1)***

$t = \pm\sqrt{m}$ (seen anywhere) ***A1***

substitutes one of their $t = \sqrt{m}$ **OR** $t = -\sqrt{m}$ into $c = \frac{1}{t} - t^3$ **OR** $c = \frac{1}{t}\left(1 - t^4\right)$ ***M1***

$c = \pm\frac{1}{\sqrt{m}}\left(1 - m^2\right)$ ***A1***

so either $c = \frac{1}{\sqrt{m}}\left(1 - m^2\right)$ **OR** $c = \frac{1}{\sqrt{m}}\left(m^2 - 1\right)$ ***AG***

[4 marks]

(b.ii) Determine the set of values of m for which there exists at least one line normal to H. [1]

Markscheme

$m \in \mathbb{R}^+$ (or equivalent) ***A1***

[1 mark]

(c) Hence, or otherwise, determine the set of values of m for which there exists exactly

(c.i) one line normal to H; [1]

Markscheme

$m = 1$ ***A1***

[1 mark]

(c.ii) two lines normal to H. [2]

Markscheme

$m \in \mathbb{R}^+, m \neq 1$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $0 < m < 1$ and ***A1*** for $m > 1$.

[2 marks]

(d) On the same set of axes, sketch

(d.i) the curve H; [1]

Markscheme

a correct sketch of $y = \frac{1}{x}$ ***A1***

[1 mark]

(d.ii) for an appropriate value of m, two lines that satisfy the result found in part (c)(ii). Clearly indicate the point at which each line is normal to H.

You are **not** required to state the equations of these lines nor determine where they intersect H or the coordinate axes. [2]

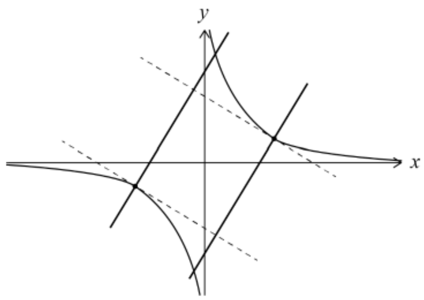
Markscheme

a line normal to H such that $c \neq 0$ ***A1***

a second parallel line which is normal to H such that $c \neq 0$ ***A1***

Note: For normal lines that cross the y-axis, award ***A1A0*** if the y-axis intercepts are not approximately equidistant from O.

e.g.



Note: Accept correct sketches that show the point(s) where the line(s) is/are normal. Candidates are not required to show dashed lines as shown above.
Award a maximum ***A1A0*** for a normal line sketched for $0 < m < 1$ and a normal line sketched for $m > 1$.

[2 marks]

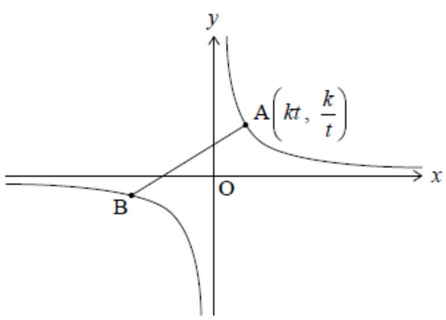
The curve F has equation $y = \frac{k^2}{x}$ where $x \in \mathbb{R},\ x \neq 0$ and $k \in \mathbb{R},\ k \neq 0$.

The point $\mathrm{A}\left(kt,\ \frac{k}{t}\right)$, where $t \in \mathbb{R},\ t \neq \pm 1$, lies on F.

The line normal to F at A intersects F again at point B.

The line segment $[\mathrm{AB}]$ is shown in the following diagram.

diagram not to scale



(e) The equation of the line normal to F at A is given by $y = t^2x - kt^3 + \frac{k}{t}$.

(e.i) Show that the x-coordinates of A and B satisfy the quadratic equation

$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0.$ [3]

Markscheme

$\frac{k^2}{x} = t^2x - kt^3 + \frac{k}{t}$ ***(A1)***

attempts to form a quadratic equation in x^2 ***M1***

eg. $t^2x^2 - kt^3x + \frac{k}{t}x = k^2$ **OR** $x^2 - ktx + \frac{k}{t^3}x - \frac{k^2}{t^2} = 0$ ***A1***

OR $\frac{k^2}{t^2} = x^2 - \left(kt - \frac{k}{t^3}\right)x$ **OR** $t^2x^2 - k\left(t^3 - \frac{1}{t}\right)x - k^2 = 0$

$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0$ ***AG***

Note: Award ***(A0)M1A0*** for attempting to verify that kt is a root of the ***AG*** (or its equivalent quadratic equation).
Award ***(A0)M0A0*** for showing that A lies on F.

[3 marks]

(e.ii) Hence, by considering either the sum or product of the roots of this quadratic equation, or otherwise, determine the coordinates of B. [3]

Markscheme

METHOD 1

let α be the x-coordinate of A and let β be the x-coordinate of B

EITHER

recognizes that the sum of roots is $k\left(t-\frac{1}{t^3}\right)$ ***(M1)***

Note: Award ***(M1)*** for $-k\left(t-\frac{1}{t^3}\right)$.

$kt+\beta=k\left(t-\frac{1}{t^3}\right)$

OR

recognizes that the product of roots is $-\frac{k^2}{t^2}$ ***(M1)***

Note: Award ***(M1)*** for $\frac{k^2}{t^2}$.

$kt\beta=-\frac{k^2}{t^2}$

THEN

$(\beta=)-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},\ -kt^3\right)$

METHOD 2

recognizes that $(x-kt)$ is a factor of $x^2-k\left(t-\frac{1}{t^3}\right)x-\frac{k^2}{t^2}(=0)$ ***(M1)***

$(x-kt)\left(x+\frac{k}{t^3}\right)(=0)$

$\Rightarrow x=kt,-\frac{k}{t^3}$

$-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},\ -kt^3\right)$

METHOD 3

attempts to use the quadratic formula ***(M1)***

$x = \frac{k\left(t-\frac{1}{t^3}\right)\pm\sqrt{\left(k\left(t-\frac{1}{t^3}\right)\right)^2+4\left(\frac{k^2}{t^2}\right)}}{2}$

$\Rightarrow x = kt, -\frac{k}{t^3}$

$-\frac{k}{t^3}$ is the other root ***A1***

$y = -kt^3$ ***A1***

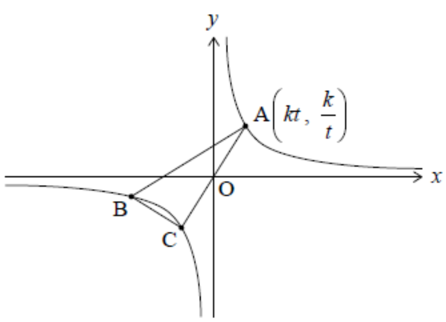
coordinates of B are $\left(-\frac{k}{t^3},\ -kt^3\right)$

[3 marks]

From A, the line passing through the origin O intersects F again at point C.

Points A, B and C form triangle ABC as shown in the following diagram.

diagram not to scale



(f) Prove that $\mathrm{B\hat{C}A}$ is a right angle. [6]

Markscheme

METHOD 1

$\mathrm{A}\left(kt,\ \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3},\ -kt^3\right)$

$\mathrm{C}\left(-kt,\ -\frac{k}{t}\right)$ ***(A1)***

attempts to find their m_{AC} or m_{BC} ***(M1)***

$m_{\mathrm{AC}} = \frac{1}{t^2}$ **AND** $m_{\mathrm{BC}} = \frac{-\frac{k}{t}-\left(-kt^3\right)}{-kt-\left(-\frac{k}{t^3}\right)} = -t^2\ \left(= \frac{kt^3-\frac{k}{t}}{\frac{k}{t^3}-kt} = -t^2\right)\ (t \neq \pm 1)$ (or equivalent) ***A1***

attempts to find their $m_{\mathrm{AC}} \times m_{\mathrm{BC}}$ **OR** recognizes that $m_{\mathrm{AC}} = -\frac{1}{m_{\mathrm{BC}}}$ (or equivalent) ***(M1)***

$= \frac{1}{t^2} \times \left(-t^2\right)$ **OR** eg. $\frac{1}{t^2} \times \frac{kt^4-k}{t} \times \frac{t^3}{k-kt^4} = \frac{t^3}{t^3} \times \frac{kt^4-k}{-(kt^4-k)}$ **OR** $-\frac{1}{\frac{1}{t^2}} = -t^2$ (or equivalent) ***A1*** $= -1$ and so $[\mathrm{BC}] \perp [\mathrm{CA}]$ ***R1*** hence $\mathrm{B\widehat{C}A}$ is a right angle ***AG***

METHOD 2

$\mathrm{A}\left(kt,\ \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3},\ -kt^3\right)$

$\mathrm{C}\left(-kt,\ -\frac{k}{t}\right)$ ***(A1)***

$m_{\mathrm{AC}} = \frac{1}{t^2}$ ***A1***

attempts to find their equation of the perpendicular to $[\mathrm{AC}]$ at C ***(M1)***

$y - \left(-\frac{k}{t}\right) = -t^2(x - (-kt))$

substitutes $x = -\frac{k}{t^3}$ into their equation of the perpendicular to $[\mathrm{AC}]$ at C ***(M1)***

$y = -kt^3$ ***A1***

B lies on the perpendicular to $[\mathrm{AC}]$ at C $(\mathrm{B} \neq \mathrm{C})$ and so $[\mathrm{BC}] \perp [\mathrm{CA}]$ ***R1***

hence $\mathrm{B\widehat{C}A}$ is a right angle ***AG***

METHOD 3

$\mathrm{A}\left(kt,\ \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3},\ -kt^3\right)$

$\mathrm{C}\left(-kt,\ -\frac{k}{t}\right)$ **OR** $\overrightarrow{\mathrm{OC}} = \begin{pmatrix} -kt \\ -\frac{k}{t} \end{pmatrix}$ ***(A1)***

attempts to find their $\overrightarrow{\mathrm{CA}}$ **OR** their $\overrightarrow{\mathrm{CB}}$ ***(M1)***

$\overrightarrow{\mathrm{CA}} = \begin{pmatrix} 2kt \\ \frac{2k}{t} \end{pmatrix}$ **AND** $\overrightarrow{\mathrm{CB}} = \begin{pmatrix} -\frac{k}{t^3} + kt \\ -kt^3 + \frac{k}{t} \end{pmatrix}$ ***A1***

attempts to find their $\overrightarrow{\mathrm{CA}} \cdot \overrightarrow{\mathrm{CB}}$ ***(M1)***

$\overrightarrow{\mathrm{CA}} \cdot \overrightarrow{\mathrm{CB}} = 2kt\left(-\frac{k}{t^3} + kt\right) + \frac{2k}{t}\left(-kt^3 + \frac{k}{t}\right)$

$= -\frac{2k^2}{t^2} + 2k^2t^2 - 2k^2t^2 + \frac{2k^2}{t^2}$ ***A1***

$= 0$ (and since $t \neq \pm 1$, $\left|\overrightarrow{\mathrm{CB}}\right| \neq 0$) ***R1***

then $\mathrm{B\widehat{C}A}$ is a right angle ***AG***

Note: Award as above if $\overrightarrow{\mathrm{OA}} = \begin{pmatrix} kt \\ \frac{k}{t} \end{pmatrix}$ is used instead of $\overrightarrow{\mathrm{CA}} = \begin{pmatrix} 2kt \\ \frac{2k}{t} \end{pmatrix}$.

METHOD 4

$\mathrm{A}\left(kt,\ \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3},\ -kt^3\right)$

$\mathrm{C}\left(-kt,\ -\frac{k}{t}\right)$ ***(A1)*** attempts to find one of their AB^2, AC^2 or BC^2 ***(M1)***

EITHER

$\mathrm{AB}^2 = \left(kt - \left(-\frac{k}{t^3}\right)\right)^2 + \left(\frac{k}{t} - \left(-kt^3\right)\right)^2$ ***A1***

OR

$\mathrm{AC}^2 = (kt - (-kt))^2 + \left(\frac{k}{t} - \left(-\frac{k}{t}\right)\right)^2$ ***A1***

OR

$\mathrm{BC}^2 = \left(-kt - \left(\frac{k}{t^3}\right)\right)^2 + \left(-\frac{k}{t} - \left(-kt^3\right)\right)^2$ ***A1***

THEN

attempts to show that their $\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{BC}^2$ ***(M1)***

$\mathrm{AB}^2 = \frac{k^2}{t^6}\left(t^4+1\right)^3$ (or equivalent) **AND** $\mathrm{AC}^2 + \mathrm{BC}^2 = \frac{k^2}{t^6}\left(t^4+1\right)^3$ (or equivalent) ***A1***

$\mathrm{AB}^2 = k^2t^6 + 3k^2t^2 + \frac{3k^2}{t^2} + \frac{k^2}{t^6}$ **AND** $\mathrm{AC}^2 + \mathrm{BC}^2 = k^2t^6 + 3k^2t^2 + \frac{3k^2}{t^2} + \frac{k^2}{t^6}$

$\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{BC}^2$ (and so, by the converse of Pythagoras' theorem) ***R1***

$\mathrm{B\widehat{C}A}$ is a right angle ***AG***

[6 marks]

10. [Maximum mark: 29] 25N.3.AHL.TZ1.2

In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population, x, as a function of t, where t is the time, in years, since the area became protected.

(a) Initially, the researchers consider using the logistic model

$x = \frac{L}{1+C\mathrm{e}^{-kt}}$, where $L, C, k \in \mathbb{R}^+$.

The researchers decide to let $L = 200$.

(a.i) State the assumption being made in assuming $L = 200$. [1]

Markscheme

that the stable/long term population will be the same as before the reduction. ***A1***

Note: Condone 'the maximum value / carrying capacity of the population will be 200'.

[1 mark]

At $t = 0$, the population of wolves is 40.

(a.ii) Find the value of C. [2]

Markscheme

attempt to substitute into logistic function ***(M1)***

$40 = \frac{200}{1+Ce^0}$

$40 + 40C = 200$

$C = 4$ ***A1***

[2 marks]

At $t = 5$, the population of wolves is found to have increased to 70.

(a.iii) Find the value of k. [2]

Markscheme

correct substitution into their logistic function ***(A1)***

$70 = \frac{200}{1+4e^{-5k}}$

$k = 0.153\ \left(0.153451\ldots,\ -\frac{1}{5}\ln\left(\frac{130}{280}\right)\right)$ ***A1***

[2 marks]

(a.iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

Markscheme

correct substitution into their logistic function ***(A1)***

$x = \frac{200}{1+4e^{-10\times0.153451\ldots}} (= 107.397\ldots)$

$x = 107$ ***A1***

[2 marks]

(b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$\frac{dx}{dt} = ax\ln\left(\frac{200}{x}\right), a \in \mathbb{R}^+.$

(b.i) Write down the value of $\frac{dx}{dt}$ when $x = 200$. [1]

Markscheme

$\frac{dx}{dt} = 0$ ***A1***

[1 mark]

(b.ii) Interpret your answer to part (b)(i) in context. [1]

Markscheme

the population is steady/stable (at 200) **OR** this (200) is the equilibrium population ***A1***

Note: Accept phrases such as 'the rate of change of wolf population is zero'. Do not accept 200 is the maximum. Do not follow through from an incorrect (b)(i).

[1 mark]

Consider the function $f(x) = \ln\left(\ln 200 - \ln x\right)$, where $0 < x < 200$.

(b.iii) Show that $f'(x) = \frac{-1}{x \ln\left(\frac{200}{x}\right)}$. [2]

Markscheme

EITHER

$\frac{1}{\ln 200 - \ln x} \times$ an attempt to differentiate $\ln 200 - \ln x$ ***M1***

$f'(x) = \frac{1}{\ln 200 - \ln x} \times \left(-\frac{1}{x}\right)$ ***A1***

OR

$\frac{1}{\ln\left(\frac{200}{x}\right)} \times \frac{x}{200} \times -\frac{200}{x^2}$ ***M1A1***

Note: Award ***M1*** if there is a product of three expressions and two are correct.

$= \frac{-1}{x \ln\left(\frac{200}{x}\right)}$ ***AG***

[2 marks]

(b.iv) Hence, use separation of variables to show that the general solution of

$\frac{\mathrm{d}x}{\mathrm{d}t} = ax \ln\left(\frac{200}{x}\right)$, where $0 < x < 200$,

can be written as

$\ln x = \ln 200 - A\mathrm{e}^{-at}$,

where A is an arbitrary positive constant. [5]

Markscheme

attempt to separate variables ***M1***

$\int \frac{-1}{x \ln\left(\frac{200}{x}\right)} \,\mathrm{d}\,x = \int -a \,\mathrm{d}\,t$ ***A1***

Note: Award ***A1*** also for $\int \frac{1}{x\left(\ln\left(\frac{200}{x}\right)\right)} \mathrm{d}\,x = \int a \mathrm{d}\,t$.

$\ln(\ln 200 - \ln x) = -at(+c)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

$\ln 200 - \ln x = \mathrm{e}^{-at+c}$ **OR** $\ln 200 - \ln x = A\mathrm{e}^{-at}$, $(A = \mathrm{e}^{c})$ ***A1***

$\ln x = \ln 200 - A\mathrm{e}^{-at}$ ***AG***

[5 marks]

(b.v) Use the size of the wolf population at $t = 0$ to find the value of A. Give your answer in the form $A = \ln p$, where $p \in \mathbb{Z}^{+}$. [2]

Markscheme

correct substitution $\ln 40 = \ln 200 - A\mathrm{e}^{-0}$ ***(A1)***

$A = \ln(5)$ ***A1***

$(p = 5)$

Note: Award ***A1A0*** for an unsupported 1.61 $(1.60943\ldots)$.

[2 marks]

(b.vi) Use the size of the wolf population at $t = 5$, given in part (a), to show that $a = 0.0855$, correct to three significant figures. [2]

Markscheme

correct substitution $\ln 70 = \ln 200 - \ln 5 \times \mathrm{e}^{-5a}$ ***A1***

$a = 0.0854528\ldots$ ***A1***

Note: Award ***A1*** if their unrounded answer has at least four significant figures and rounds to 0.0855 **OR** if an exact value equivalent to $0.2 \ln\left(\frac{\ln 5}{\ln\left(\frac{200}{70}\right)}\right)$ is seen.

$a = 0.0855$ to 3 significant figures ***AG***

[2 marks]

(b.vii) Use the Gompertz model to predict the size of the wolf population at $t = 10$. Give your answer correct to the nearest whole number. [3]

Markscheme

correct substitution $\ln x = \ln 200 - \ln 5 \times \mathrm{e}^{-10\times 0.0854528\ldots}$ ***(A1)***

$100.839\ldots$ ***(A1)***

Note: Accept $100.875\ldots$ from use of 3 sf 0.0855.

101 to the nearest whole number ***A1***

[3 marks]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

Markscheme

both are overestimates **OR** the Gompertz is closer to the true value ***A1***

Note: Follow through on their answers for 2(a)(iv) and 2(b)(vii) if both positive values and these values are compared with 85.

[1 mark]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.0855x\ln\left(\frac{200}{x}\right) - 0.03x$.

(d.i) Use Euler's method, with a step size of 0.5 years and an initial value of $x_0 = 70$ when $t = 5$, to find an estimate for the size of the wolf population when $t = 10$. Give your answer correct to the nearest whole number. [4]

Markscheme

attempt to use Euler ***(M1)***

Note: Condone maximum of one error from missing x_n+ at start, or incorrect/missing subscripts, absence of 0.5.

$x_{n+1} = x_n + 0.5\left(0.0855x_n\ln\left(\frac{200}{x_n}\right) - 0.03x_n\right)$ ***(A1)***

evidence of correct use of Euler's method ***(A1)***

Note: Award ***(A1)*** for any correct value of x_{n+1} for $n = 2, 3, \ldots$ seen (including this intermediate value presented as a final answer).

after 10 years the population will be 89 wolves $(89.42803\ldots)$ ***A1***

[4 marks]

(d.ii) Comment on your answer. [1]

Markscheme

correctly compare their result to the actual number of wolves or to the other models or both. Allow even if this is implicit, for example, it is more accurate / realistic than the other models ***A1***

[1 mark]

11. [Maximum mark: 26] 25N.3.AHL.TZ3.1

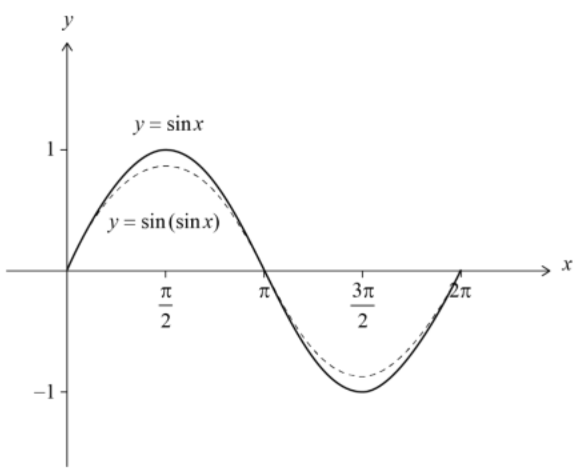
The following question explores features of composed trigonometric functions, such as $\sin(\sin x), \sin(\sin(\sin x))$.

Suppose $S_n(x)$ denotes the function $\sin x$ composed within itself $n-1$ times, defined for $n \geq 1, n \in \mathbb{Z}^+$, where $0 \leq x \leq 2\pi$.

For example, $S_1(x) = \sin x$ and $S_2(x) = \sin(\sin x)$ where $0 \leq x \leq 2\pi$.

(a) On the same axes, sketch and label the graphs of $y = S_1(x)$ and $y = S_2(x)$. On your sketch, show the values of the intercepts with the axes. [4]

Markscheme



correct shape $y = \sin x$ ***A1***

correct domain ***A1***

intercepts on x-axis at $0, \pi, 2\pi$ ***A1***

correct shape $y = \sin(\sin x)$ with a smaller maximum value and a larger minimum value than $y = \sin x$ and the same x-intercepts as $y = \sin x$ ***A1***

Note: For candidates who do not label their graphs with $\sin x$ and $\sin(\sin x)$ award at most ***A1A1A1A0***.
For candidates who sketch their graphs on separate axes award at most ***A1A1A1A0***.
These marks are independent of each other.

[4 marks]

(b) Determine the maximum value of

(b.i) $S_1(x)$; [1]

Markscheme

1 ***A1***

[1 mark]

(b.ii) $S_2(x)$; [1]

Markscheme

$0.841\ (0.841470\ldots)$ ***A1***

[1 mark]

(b.iii) $S_3(x)$. [1]

Markscheme

$0.746\ (0.745624\ldots)$ ***A1***

[1 mark]

(c) Find the least value of n for which the maximum value of $S_n(x)$ is less than 0.6. [3]

Markscheme

use of GDC eg tables (to vary the value of n) ***(M1)***

$\left(S_4\left(\frac{\pi}{2}\right) = 0.678430\ldots,\ S_5\left(\frac{\pi}{2}\right) = 0.627571\ldots,\ S_6\left(\frac{\pi}{2}\right) = 0.587180\ldots\right)$

$n = 6$ ***A2***

[3 marks]

Consider the graph of $y = S_2(x)$.

(d) By considering the equation $\frac{dy}{dx} = 0$, show that there are exactly two points of zero gradient, one at $x = \frac{\pi}{2}$ and one at $x = \frac{3\pi}{2}$. [6]

Markscheme

attempt to find $\frac{dy}{dx}$ using chain rule ***(M1)***

$\left(\frac{dy}{dx} =\right) \cos(\sin x)\cos x$ ***A1***

$\cos(\sin x)\cos x = 0$

considers $\cos x = 0$ ***M1***

$x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ ***A1***

considers $\cos(\sin x) = 0$ ***M1***

$\sin x = \frac{\pi}{2} \Rightarrow$ no other solution for x ***R1***

Note: The ***M1R1*** above are independent of the previous ***M1A1***.

hence curve has exactly two points of zero gradient at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ ***AG***

[6 marks]

The derivative $S'_n(x) = \frac{d}{dx}(S_n(x))$ can be expressed as a product of cosine functions, as follows:

$S\prime_n(x) = \cos(S_{n-1}(x))\cos(S_{n-2}(x))\ldots\ \cos(S_1(x))\cos x.$

(e) Hence, show that $S\prime_3(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x.$ [1]

Markscheme

$S\prime_3(x) = \cos(S_2(x))\cos(S_1(x))\cos x$ ***A1***

$S\prime_3(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$ ***AG***

[1 mark]

(f) Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$

$S\prime_n(x) = \cos(S_{n-1}(x))\cos(S_{n-2}(x))\ldots\ \cos(S_1(x))\cos x.$ [6]

Markscheme

consider case when $n = 1$ $\text{LHS} = \frac{d}{dx}(\sin x) = \cos x = \text{RHS}$ ***A1***

Note: Subsequent marks after this ***A1*** are independent of this mark and can be awarded.

assume true for some k, ***M1***

i.e. $\frac{d}{dx}(S_k(x)) = \cos(S_{k-1}(x))\cos(S_{k-2}(x))\ldots\ \cos(S_1(x))\cos x$

Note: Award ***M0*** for statements such as "let $n = k$" or "assume that $n = k$ is true". The assumption of truth of the statement must be clear. Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

EITHER

recognition that $S_{k+1}(x) = \sin(S_k(x))$ ***M1***

$\frac{d}{dx}(S_{k+1}(x)) = \frac{d}{dx}\sin(S_k(x))$

$= \cos(S_k(x))\frac{d}{dx}(S_k(x))$ ***A1***

$= \cos(S_k(x))\cos(S_{k-1}(x))\cos(S_{k-2}(x))\ldots\ \cos x$ (by assumption) ***A1***

OR

recognition that $S_{k+1}(x) = S_k(\sin x)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}x}(S_{k+1}(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(S_k(\sin x))$

$= S\prime_k(\sin x)\cos x$ ***A1***

$= \cos(S_{k-1}(\sin x))\cos(S_{k-2}(\sin x))\ldots\cos(\sin x)\cos x$ (by assumption)

$= \cos(S_k(x))\cos(S_{k-1}(x))\cos(S_{k-2}(x))\ldots\cos x$ ***A1***

THEN

hence true for case $n = k+1$

true for the case $n = 1$, when case $n = k$ assumed true, case $n = k+1$ proven to be true. Hence true for all positive integers n. ***R1***

Note: Only award **R1** if at least 3 of the previous 5 marks have been awarded.

[6 marks]

(g) Use l'Hôpital's rule to show that $\lim\limits_{x\to 0}\frac{S_n(x)}{x} = 1$, for $n \in \mathbb{Z}^+$. [3]

Markscheme

attempt to differentiate numerator and denominator ***(M1)***

$\lim\limits_{x\to 0}\frac{S_n(x)}{x} = \lim\limits_{x\to 0}\frac{\cos(S_{n-1}(x))\cos(S_{n-2}(x))\ldots\cos(S_1(x))\cos x}{1}$ ***A1***

$= \frac{\cos(0)\cos(0)\ldots\cos(0)\cos(0)}{1}$ ***A1***

$= 1$ ***AG***

Note: For candidates who don't include the limit award a maximum of ***(M1)A1A0***.

[3 marks]

12. [Maximum mark: 29] 25N.3.AHL.TZ3.2

The following question uses Maclaurin series to investigate approximations of mathematical constants and the accuracies of such approximations.

(a) Given $|x| < 1$, find the sum to infinity of the geometric series $1 - x^2 + x^4 - x^6 + \ldots$. [2]

Markscheme

attempt to use $S_\infty = \frac{u_1}{1-r}$ ***(M1)***

$= \frac{1}{1+x^2}$ ***A1***

Note: Allow working with $u_1 = 1$ and $r = -x^2$ or $u_1 = 1 - x^2$ and $r = x^4$.

[2 marks]

(b) Hence, use integration to show that the Maclaurin series of $\arctan x$ may be expressed as $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$. [3]

Markscheme

$\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x (+c)$ ***A1***

$\int (1 - x^2 + x^4 - x^6 + \ldots) \mathrm{d}x = (c+)x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots (= \arctan x)$ ***A1***

Note: The ***A1*** marks above are independent.

when $x = 0, \arctan 0 = c + 0 \Rightarrow c = 0$ ***R1***

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***AG***

[3 marks]

(c) Using $x = \frac{1}{\sqrt{3}}$ and the first **three** (non-zero) terms of the Maclaurin series of $\arctan x$, find an approximation for π to three decimal places. [3]

Markscheme

$\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6} \left(\approx \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 \right)$ ***A1***

attempt to evaluate $\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5$ ***M1***

$\pi \approx 3.156$ ***A1***

[3 marks]

The Maclaurin series of $\arctan x$ is an example of an alternating series, *ie* a series where consecutive terms are positive and negative. Consider the following theorem.

Theorem: For alternating series with terms of decreasing magnitude, the error obtained in using a finite number of terms is less than or equal to the absolute value of the next term in the sequence.

Using the theorem, the maximum error in using the first three (non-zero) terms as an approximation to $\arctan x$ is given by $\left| -\frac{x^7}{7} \right|$. In other words, $\left| \arctan x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} \right) \right| \le \left| -\frac{x^7}{7} \right|$.

(d) Determine how many (non-zero) terms of the series would need to be used, such that the error in approximating $\arctan\left(\frac{1}{\sqrt{3}}\right)$ is less than 0.0001. [3]

Markscheme

METHOD 1

recognition that in the Maclaurin expansion of $\arctan x$, the error term is $\frac{x^{2m-1}}{2m-1}$ ***(M1)***

attempt to solve $\frac{1}{2m-1} \times \left(\frac{1}{\sqrt{3}}\right)^{2m-1} < 0.0001$ **OR** $m = 6.60583\ldots (= 7)$ ***M1***

6 (non-zero) terms are needed *A1*

Note: Accept the error term in an absolute value.
Give marks as above for candidates who use $2m+1$ instead of $2m-1$. This gives $m = 5.60583\ldots (=6)$ and leads to needing 6 terms.
Give marks as above for candidates who use n as the exponent and divisor. This gives $n = 12.211675\ldots$ and leads to needing 6 terms.

METHOD 2

recognition that in the Maclaurin expansion of $\arctan x$, the error term is $\frac{x^{2m-1}}{2m-1}$ *(M1)*

attempt to find the value of the error term for at least two consecutive values of m *M1*

error terms are $\frac{\left(\frac{1}{\sqrt{3}}\right)^7}{7} = 0.003054\ldots$, $\frac{\left(\frac{1}{\sqrt{3}}\right)^9}{9} = 0.0007919\ldots$, $\frac{\left(\frac{1}{\sqrt{3}}\right)^{11}}{11} = 0.0002159\ldots$, $\frac{\left(\frac{1}{\sqrt{3}}\right)^{13}}{13} = 0.00006092\ldots$

6 (non-zero) terms are needed *A1*

Note: Accept the error term in an absolute value.
Accept $2m+1$ or n in place of $2m-1$.

METHOD 3

attempt to find the actual error using $\arctan x -$ (first m non-zero terms) *(M1)*

attempt to find the error for at least two consecutive values of m *M1*

$(m = 4 \Rightarrow 0.000623, \quad m = 5 \Rightarrow 0.000169, \quad m = 6 \Rightarrow 0.0000473)$

6 (non-zero) terms are needed *A1*

[3 marks]

(e) By using integration by parts, show that $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}\, x = \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}$. [4]

Markscheme

$u = \arctan x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$ (or equivalent) *(A1)*

$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}\, x = [x \arctan x]_0^{\frac{1}{\sqrt{3}}} - \int_0^{\frac{1}{\sqrt{3}}} \frac{x}{1+x^2} \, \mathrm{d}\, x$ *A1A1*

Note: Award *A1* for each term.

$= [x \arctan x]_0^{\frac{1}{\sqrt{3}}} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^{\frac{1}{\sqrt{3}}} \left(= \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}\right)$ *A1*

$= \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}$ *AG*

[4 marks]

(f) Determine the value of $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) \mathrm{d}\, x$, giving your answer to six decimal places. [2]

Markscheme

$\left(\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) \mathrm{d}\, x = \right) 0.158422$ ***A2***

[2 marks]

(g) Use the results in parts (e) and (f) to find an approximation for π. Give your answer to four decimal places. [2]

Markscheme

attempt to solve (for π) $\frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3} \approx 0.158422$ ***M1***

$\pi \approx 3.1412$ ***A1***

[2 marks]

$\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right) \mathrm{d}\, x$ may be considered as the sum of alternating terms.

Hence, $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right) \mathrm{d}\, x = \int_0^{\frac{1}{\sqrt{3}}} x \,\mathrm{d}\, x + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^3}{3}\right) \mathrm{d}\, x + \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^5}{5}\right) \mathrm{d}\, x + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^7}{7}\right) \mathrm{d}\, x + \ldots$.

$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \,\mathrm{d}\, x$ is approximated using the sum of the first four definite integrals.

(h) Verify that the theorem used in part (d) holds in this case. [5]

Markscheme

theorem error is given by $\left|\int_0^{\frac{1}{\sqrt{3}}} \frac{x^9}{9} \,\mathrm{d}\, x\right| \left(= \left[\frac{x^{10}}{90}\right]_0^{\frac{1}{\sqrt{3}}}\right)$ ***(M1)***

$= 4.57 \times 10^{-5}$ ***A1***

attempt to evaluate their error ***(M1)***

$\left(\frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}\right) - 0.158422$ **OR** $0.158458558\ldots - 0.158422$

$= 3.69 \times 10^{-5}$ ***A1***

Note: Accept $= 3.73 \times 10^{-5}$ from using the unrounded answer from part (f).
The ***(M1)A1*** above are independent of the previous ***(M1)A1.***

$3.69 \times 10^{-5} < 4.57 \times 10^{-5}$ ***R1***

hence theorem holds ***AG***

[5 marks]

Suppose that the maximum error in approximating $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}\, x$ is required to be at most 1×10^{-6}.

(i) Determine the smallest number of (non-zero) terms of the Maclaurin series for $\arctan x$ that should be used. [5]

Markscheme

METHOD 1

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^n}{n}\right) \mathrm{d}\, x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\frac{\left(\frac{1}{\sqrt{3}}\right)^{n+1}}{n(n+1)} < 1 \times 10^{-6}$ ***A1***

attempt to solve using GDC ***(M1)***

$n = 14.3336\ldots$

$\Rightarrow n = 15$ ***A1***

(hence 8^{th} term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 2

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \frac{x^{2m-1}}{2m-1} \mathrm{d}\, x$ **OR** $\int_0^{\frac{1}{\sqrt{3}}} \frac{x^{2m+1}}{2m+1} \mathrm{d}\, x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\frac{\left(\frac{1}{\sqrt{3}}\right)^{2m}}{2m(2m-1)} < 1 \times 10^{-6}$ **OR** $\frac{\left(\frac{1}{\sqrt{3}}\right)^{2m+2}}{(2m+1)(2m+2)} < 1 \times 10^{-6}$ ***A1***

attempt to solve using GDC ***(M1)***

$\Rightarrow m = 7.66681\ldots$ **OR** $\Rightarrow m = 6.66681\ldots$ ***A1***

(hence 8^{th} term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 3

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^n}{n}\right) \mathrm{d}\, x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{13}}{13}\right) \mathrm{d}\,x = 2.51234\ldots \times 10^{-6}$ **OR** $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{15}}{15}\right) \mathrm{d}\,x = 6.35065\ldots \times 10^{-7}$ ***A1***

attempt to find the value of the integral for at least two consecutive odd values of $n, n > 9$ ***(M1)***

$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{13}}{13}\right) \mathrm{d}\,x = 2.51234\ldots \times 10^{-6}$, $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{15}}{15}\right) \mathrm{d}\,x = 6.35065\ldots \times 10^{-7}$ ***A1***

Note: Award this ***A1*** for the second correct value.

(hence 8^{th} term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 4

attempt to find the actual error using $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \,\mathrm{d}\,x -$(first m non-zero terms) ***(M1)***

error with 6 terms is $2.00830\ldots \times 10^{-6}$ **OR** error with 7 terms is $5.04041\ldots \times 10^{-7}$ ***A1***

attempt to find the error for at least two consecutive values of m ***(M1)***

error with 6 terms is $2.00830\ldots \times 10^{-6}$ and error with 7 terms is $5.04041\ldots \times 10^{-7}$ ***A1***

Note: Award this ***A1*** for the second correct value.

we require 7 (non-zero) terms ***A1***

[5 marks]

13. [Maximum mark: 23] 25M.3.AHL.TZ1.1

This question asks you to use polynomial functions to model some situations in probability.

Two unbiased tetrahedral (four-sided) dice with faces labelled $1,\ 2,\ 3$ and 4 are thrown and the scores recorded.

The random variable M denotes the maximum of these two scores.

The probability distribution of M is given in the following table.

m	1	2	3	4
$\mathrm{P}(M=m)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

(a) Find $\mathrm{E}(M)$. [2]

Markscheme

attempts to use $\mathrm{E}(M) = \sum m\mathrm{P}(M = m)$ ***(M1)***

$= 1 \times \frac{1}{16} + 2 \times \frac{3}{16} + 3 \times \frac{5}{16} + 4 \times \frac{7}{16}$ (or equivalent)

$= \frac{25}{8} \left(= 3.125, \ = \frac{50}{16}, \ = 3.13\right)$ ***A1***

[2 marks]

An alternative way to represent the probability distribution of M is to use a polynomial function, G, where $G(t) = \sum_{\mathrm{m}=1}^{4} \mathrm{P}(M = m)t^m$.

Hence, for the distribution of $M, G(t) = \frac{1}{16}t + \frac{3}{16}t^2 + \frac{5}{16}t^3 + \frac{7}{16}t^4$.

(b) Find $G(1)$. [1]

Markscheme

$G(1) = 1$ ***A1***

[1 mark]

(c.i) Find $G\prime(t)$. [2]

Markscheme

$G\prime(t) = \frac{1}{16} + \frac{3}{8}t + \frac{15}{16}t^2 + \frac{7}{4}t^3 \ \left(= \frac{1}{16} + \frac{6}{16}t + \frac{15}{16}t^2 + \frac{28}{16}t^3\right)$ ***A2***

Note: Award ***A1A0*** for three correct terms.

[2 marks]

(c.ii) Hence, show that $G\prime(1) = \mathrm{E}(M)$. [3]

Markscheme

substitutes $t = 1$ into their $G\prime(t)$ ***(M1)***

$G\prime(1) = \frac{1}{16} + \frac{3}{8} + \frac{15}{16} + \frac{7}{4} \ \left(= \frac{1}{16} + \frac{6}{16} + \frac{15}{16} + \frac{28}{16}\right)$ ***A1***

$= \frac{25}{8} \left(= 3.125, \ = \frac{50}{16}\right)$ ***A1***

$= \mathrm{E}(M)$ ***AG***

Note: Award a maximum of ***M1A1A0FT*** for an incorrect $\mathrm{E}(M)$ obtained with an incorrect $G\prime(t)$.
Award a maximum of ***M1A1A0FT*** for a correct $\mathrm{E}(M)$ obtained with an incorrect $G\prime(t)$.

[3 marks]

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable X denotes the total number of red balls selected.

The probability distribution of X can be represented by the polynomial function, G_X, where

$G_x(t) = \sum_{x=0}^{2} \mathrm{P}(X = x)t^x$.

(d) Show that $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$, making it clear how the coefficients of $G_X(t)$ have been determined. [5]

Markscheme

attempts to find at least one required probability ***(M1)***

$\mathrm{P}(X = 0) = \frac{3}{5} \times \frac{2}{4}$ **OR** $\mathrm{P}(X = 0) = \frac{{}^2C_0 \times {}^3C_2}{{}^5C_2} \left(= \frac{{}^3C_2}{{}^5C_2}\right)$ ***A1***

$= \frac{3}{10}$

$\mathrm{P}(X = 1) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} \left(= 2 \times \frac{6}{20}\right)$ ***A1A1***

Note: Award ***A1*** for $\frac{2}{5} \times \frac{3}{4}$ **OR** $\frac{6}{20}$ and ***A1*** for $\frac{3}{5} \times \frac{2}{4}$ **OR** $\frac{6}{20}$.
Award ***A2*** for $\mathrm{P}(X = 1) = \frac{{}^2C_1 \times {}^3C_1}{{}^5C_2} \left(= \frac{2 \times {}^3C_1}{{}^5C_2}\right)$.

$= \frac{3}{5}$

EITHER

$\mathrm{P}(X = 2) = \frac{2}{5} \times \frac{1}{4}$ **OR** $\mathrm{P}(X = 2) = \frac{{}^2C_2 \times {}^3C_0}{{}^5C_2} \left(= \frac{{}^2C_2}{{}^5C_2}, \ = \frac{{}^3C_0}{{}^5C_2}\right)$ ***A1***

OR

$\mathrm{P}(X = 2) = 1 - \left(\frac{3}{10} \times \frac{3}{5}\right)$ ***A1***

THEN

$= \frac{1}{10}$

leading to $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$ ***AG***

Note: Award ***A*** marks as appropriate for clearly labelled correct tree diagrams.
Do not award ***A*** marks for $\mathrm{P}(X = 0) = \frac{3}{10}$, $\mathrm{P}(X = 1) = \frac{3}{5}$ and $\mathrm{P}(X = 2) = \frac{1}{10}$ stated without any working shown.

[5 marks]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is p.

The random variable Y denotes the total number of tails obtained from tossing both coins.

The probability distribution of Y can be represented by the polynomial function, G_Y, where

$G_Y(t) = \sum_{y=0}^{2} \mathrm{P}(Y = y)t^y$.

(e) Given that the coefficient of t^2 in $G_Y(t)$ is $\frac{1}{3}$, find

(e.i) the value of p; [2]

Markscheme

EITHER

$\mathrm{P}(Y = 2) = \frac{1}{2}p$ ***(A1)***

OR

$\frac{1}{2}p = \frac{1}{3}$ ***(A1)***

THEN

$p = \frac{2}{3}(= 0.667)$ ***A1***

[2 marks]

(e.ii) an expression for $G_Y(t)$. [4]

Markscheme

uses their $G_Y(t) = \mathrm{P}(Y = 0)t^0 + \mathrm{P}(y = 1)t^1 + \frac{1}{3}t^2$ with their value of p ***(M1)***

$\mathrm{P}(Y = 0) = \frac{1}{6}$ ***(A1)***

$\mathrm{P}(Y = 1) = \frac{1}{2}$ ***(A1)***

$G_Y(t) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$ ***A1***

Note: Award a maximum ***(M1)(A1)(A1)A0FT*** from an incorrect value of p.
Award ***(M1)(A1)(A1)A0FT*** from $p = \frac{1}{3}$ leading to $G_Y(t) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$.

[4 marks]

The random variable Z denotes the sum of the total number of red balls selected, X, and the total number of tails obtained from tossing both coins, Y.

The probability distribution of Z can be represented by the function, G_Z, where

$G_Z(t) = G_X(t)G_Y(t)$.

(f) For random variable Z, it can be shown that $G_Z\prime(1) = \mathrm{E}(Z)$.

Use this result to find $\mathrm{E}(Z)$. [4]

Markscheme

METHOD 1

$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$ $\left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4\right)$ ***(A1)***

attempts to evaluate their $G_Z\prime(1)$ ***(M1)***

$G_Z\prime(1) = 1.96666\ldots$

$\mathrm{E}(Z) = 1.97 \ \left(= \frac{59}{30}\right)$ ***A2***

Note: Award a maximum of ***(A1)FT(M1)(M1)A0*** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).
Do not award ***(A1)FT*** if $G_Y(t)$ is not a quadratic of the required form.

METHOD 2

$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$ $\left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4\right)$ ***(A1)***

attempts to find $G_Z\prime(t)$ by differentiation ***M1***

$G_Z\prime(t) = \frac{1}{4} + \frac{5}{6}t + \frac{3}{4}t^2 + \frac{2}{15}t^3$ **OR** $G_Z\prime(t) = \left(\frac{3}{5} + \frac{1}{5}t\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right) + \left(\frac{1}{2} + \frac{2}{3}t\right)\left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)$

substitutes $t = 1$ into their $G_Z\prime(t)$ ***(M1)***

$G_Z\prime(1) = 1.96666\ldots$

$\mathrm{E}(Z) = 1.97 \ \left(= \frac{59}{30}\right)$ ***A1***

Note: Award a maximum of ***(A1)FT(M1)(M1)A0*** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).
Do not award ***(A1)FT*** if $G_Y(t)$ is not a quadratic of the required form.

METHOD 3

$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$ $\left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4\right)$ ***(A1)***

recognizes that $G_Z\prime(1) = G_X\prime(1) + G_Y\prime(1) \Rightarrow \mathrm{E}(Z) = \mathrm{E}(X) + \mathrm{E}(Y)$ ***(M1)***

attempts to evaluate their $G_X\prime(1)$ and $G_Y\prime(1)$ (or their $\mathrm{E}(X)$ or $\mathrm{E}(Y)$) ***(M1)***

$G_X\prime(1) = \frac{3}{5} + \frac{1}{5} \left(= \frac{4}{5}\right)$ and $G_Y\prime(1) = \frac{1}{2} + \frac{2}{3} \left(= \frac{7}{6}\right)$

$\mathrm{E}(Z) = 1.97 \ \left(= \frac{59}{30}\right)$ ***A1***

Note: Award a maximum of ***(A1)FT(M1)(M1)A0*** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).
Do not award ***(A1)FT*** if $G_Y(t)$ is not a quadratic of the required form.

[4 marks]

14. [Maximum mark: 32] 25M.3.AHL.TZ1.2

Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.

Consider any function f that can be differentiated twice.

The curvature, k, of any function f is defined by $k\left(x\right)=\frac{\left|f''(x)\right|}{\left(1+\left(f'(x)\right)^2\right)^{\frac{3}{2}}}$.

Consider the family of linear functions $g(x)=mx+c$, where $x\in\mathbb{R}$ and $m,\ c\in\mathbb{R}$.

(a) Show that $k\left(x\right)=0$ for this family of linear functions. [2]

Markscheme

$g'(x)=m$ **OR** $g''(x)=0$ (seen anywhere) ***(A1)***

Note: Award A0 for $g''(x)=0$ obtained from an incorrect $g'(x)$, for example, $g'(x)=1$ **OR** $g'(x)=0$.

EITHER

$k\left(x\right)=\frac{\left|0\right|}{\left(1+m^2\right)^{\frac{3}{2}}}$ **OR** $k\left(x\right)=\frac{0}{\left(1+m^2\right)^{\frac{3}{2}}}$ **OR** $k\left(x\right)=\frac{0}{\sqrt{1+3m^2+3m^4+m^6}}$ ***A1***

OR

the numerator of $k\left(x\right)$ is zero and the denominator is non-zero ***A1***

THEN

$\Rightarrow k\left(x\right)=0$ ***AG***

[2 marks]

Consider the family of quadratic functions $h(x)=ax^2+bx+c$ for $x\in\mathbb{R}$, where $a\in\mathbb{R},\ a\neq0$ and $b,\ c\in\mathbb{R}$.

For this family of quadratic functions, it is given that

$k\left(x\right)=\frac{2\left|a\right|}{\left(1+\left(2ax+b\right)^2\right)^{\frac{3}{2}}}$ and

$k'(x)=\frac{12a\left|a\right|\left(2ax+b\right)}{\left(1+\left(2ax+b\right)^2\right)^{\frac{5}{2}}}$.

The maximum value of $k\left(x\right)$ is denoted by $k_{\max}$.

(b.i) By solving $k'(x)=0$, find the value of x where $k_{\max}$ occurs.

You are **not** required to justify that this value of x leads to a maximum. [1]

Markscheme

$x = -\frac{b}{2a}$ ***A1***

Note: Do not award ***A1*** if $a = 0$ is included as a solution.

[1 mark]

(b.ii) Determine an expression for $k_{\max}$, in terms of a only. [2]

Markscheme

substitutes their value of x into $k\left(x\right)$ ***(M1)***

$k\left(-\frac{b}{2a}\right) = \frac{2|a|}{\left(1+\left(2a\left(-\frac{b}{2a}\right)+b\right)^2\right)^{\frac{3}{2}}}$ $\left(= \frac{2|a|}{\left(1+\left(-b+b\right)^2\right)^{\frac{3}{2}}}\right)$

$= 2|a|$ ***A1***

Note: Award ***(M1)A0*** if the modulus sign is not considered.

[2 marks]

(b.iii) State the value of $\lim_{x\to\infty} k\left(x\right)$ and explain briefly the significance of this result. [2]

Markscheme

EITHER

$\lim_{x\to\infty} k\left(x\right) = 0$ ***A1***

OR

the curvature (of a quadratic) approaches zero as $x \to \infty$ ***A1***

THEN

for large positive values of x, quadratic functions are close to being straight ***R1***

Note: Award ***R1*** for similar statements such as 'for large positive values of x, a quadratic function behaves like a linear function.

[2 marks]

(b.iv) Consider the quadratic functions

$p(x) = -2x^2 + 2x - 10$ where $x \in \mathbb{R}$ and

$q(x) = -2x^2 + 5x + 25$ where $x \in \mathbb{R}$.

State which one of the following statements is true and justify your answer.

A. $k_{\max}$ of $p > k_{\max}$ of q

B. $k_{\max}$ of $p < k_{\max}$ of q

C. $k_{\max}$ of $p = k_{\max}$ of q [2]

Markscheme

C ($k_{\max}$ of $p = k_{\max}$ of q) ***A1***

$k_{\max}$ of p is $(2 \times |-2| =) 4$ and $k_{\max}$ of q is $(2 \times |2| =) 4$ ***R1***

Note: Accept arguments such as \ $k_{\max} = 2|a|$ and both quadratics have $|a| = 2$.
Do not award ***A1R0***.
If modulus signs were omitted in part (b)(ii), award ***FT*** in part (b)(iv) for an answer of **B** with appropriate values shown.

[2 marks]

Consider the function $v(x) = \ln x$, where $x \in \mathbb{R},\ x > 0$.

For v, it is given that

$k(x) = \frac{x}{(1+x^2)^{\frac{3}{2}}}$ and

$k\prime(x) = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}$.

(c.i) Determine the exact value of x where $k_{\max}$ occurs.

You are **not** required to justify that this value of x leads to a maximum. [2]

Markscheme

sets $k\prime(x) = 0$ and attempts to solve for x ***(M1)***

$x^2 = \frac{1}{2}$

$x = \frac{1}{\sqrt{2}}$ ***A1***

Note: Award ***A0*** for $x = \pm\frac{1}{\sqrt{2}}\ (= \pm 0.707)$. Award ***A0*** for $x = 0.707$.

[2 marks]

(c.ii) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [4]

Markscheme

substitutes their value of x into $k(x)$ ***(M1)***

Note: Award a maximum of ***(M1)(A0)A0A0*** if their value of x is a decimal. Award ***(M1)(A0)A0A0*** for $(0.707,\ 0.385)$.

$k\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}$

Note: Award the following two marks for evaluating the above denominator.

$\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{3}{2}}$ ***(A1)***

$= \frac{3\sqrt{3}}{2\sqrt{2}}$ (or equivalent unsimplified fractional surd e.g. $\sqrt{\frac{27}{8}}$) ***A1***

$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}} \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}\right)$

$= \frac{2}{3\sqrt{3}}$ (or equivalent surd e.g. $\frac{\sqrt{12}}{9}$) ***A1***

$k_{\max} = \frac{2\sqrt{3}}{9}$ ***AG***

[4 marks]

Consider the function $w(x) = \mathrm{e}^x$, where $x \in \mathbb{R}$.

For w, it is given that

$k(x) = \frac{\mathrm{e}^x}{(1+\mathrm{e}^{2x})^{\frac{3}{2}}}$ and

$k'(x) = \frac{\mathrm{e}^x(1-2\mathrm{e}^{2x})}{(1+\mathrm{e}^{2x})^{\frac{5}{2}}}$.

(d.i) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [5]

Markscheme

EITHER

sets $k'(x) = 0$ and attempts to solve for e^{2x} ***(M1)***

$\mathrm{e}^{2x} = \frac{1}{2}$ ***A1***

$\mathrm{e}^x = \frac{1}{\sqrt{2}}$

substitutes their values for e^{2x} and e^x into $k(x)$ ***(M1)***

OR

attempts to solve $k\prime(x) = 0$ for x ***(M1)***

$x = \frac{1}{2}\ln\frac{1}{2}\ \left(= \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2\right)$ ***A1***

Note: Award a maximum of ***(M1)(A0)(M0)A0A0*** if their value of x is a decimal. Award ***(M1)(A0)(M0)A0A0*** for $(-0.347,\ 0.385)$.

substitutes their value of x into $k(x)$ ***(M1)***

THEN

for example, $\left(k\left(\ln\frac{1}{\sqrt{2}}\right) =\right)\frac{\frac{1}{\sqrt{2}}}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}\left(= \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{\frac{3}{2}}}\right)$ ***A1***

$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}}\ \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}\right)$ ***A1***

$= \frac{2}{3\sqrt{3}}$

$k_{\max} = \frac{2\sqrt{3}}{9}$ ***AG***

Note: After obtaining the penultimate ***A1*** above, candidates may state $k_{\max} = \frac{2\sqrt{3}}{9}$ as per correct work seen in part (c) (ii). In such cases, award the final ***A1.***

[5 marks]

(d.ii) Suggest a reason why v and w have the same $k_{\max}$. [1]

Markscheme

EITHER

w and v are inverse functions of each other (and as such are equally 'curved') (or equivalent) ***R1***

OR

the graphs of w and v are reflections of each other in the line $y = x$ ***R1***

OR

w and v have the same shape (or equivalent) ***R1***

[1 mark]

Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r,\ y > 0$ and r is a positive constant.

(e.i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$. [6]

Markscheme

METHOD 1

attempts to use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(-2x)\left(r^2 - x^2\right)^{-\frac{1}{2}}$ $\left(= -\frac{x}{\sqrt{r^2-x^2}}, \ = -\frac{x}{y}\right)$ ***A1***

attempts to use the quotient or product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-y+x\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2}$ $\left(= \frac{x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\right)$ ***A1***

correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ to form $= \frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$ $\left(= \frac{x}{y^2}\left(-\frac{x}{y}\right) - \frac{1}{y}\right)$ ***A1***

$= \frac{-y^2-x^2}{y^3}$ $\left(= -\frac{y^2+x^2}{y^3}\right)$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

METHOD 2

attempts to use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(-2x)\left(r^2 - x^2\right)^{-\frac{1}{2}}$ $\left(= -\frac{x}{\sqrt{r^2-x^2}}\right)$ ***A1***

attempts to use the quotient or product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***(M1)***

$= \frac{-\left(r^2-x^2\right)^{\frac{1}{2}} - x^2\left(r^2-x^2\right)^{-\frac{1}{2}}}{r^2-x^2}$ $\left(= -x^2\left(r^2 - x^2\right)^{-\frac{3}{2}} - \left(r^2 - x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

$= \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}$ $\left(= \frac{-\left(r^2-x^2\right)-x^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}, \ = -\frac{r^2}{\left(y^2\right)^{\frac{3}{2}}}\right)$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator. Candidates may express their $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of r and y instead of r and x.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

METHOD 3

attempts to use implicit differentiation on $y^2 = r^2 - x^2$ ***(M1)***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$ $\left(y\frac{\mathrm{d}y}{\mathrm{d}x} = -x\right)$ ***A1***

attempts to use the product rule and implicit differentiation ***(M1)***

$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = -1$ (or equivalent) ***A1***

correct substitution of $\frac{dy}{dx} = -\frac{x}{y}$ to form $\left(-\frac{x}{y}\right)^2 + y\frac{d^2y}{dx^2} = -1$ ***A1***

$\frac{d^2y}{dx^2} = \frac{1}{y}\left(-1 - \frac{x^2}{y^2}\right)$

$= \frac{-y^2-x^2}{y^3}$ $\left(= -\frac{y^2+x^2}{y^3}\right)$ ***A1***

$\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$ ***AG***

[6 marks]

(e.ii) Hence, show that the curvature, k, is constant for this family of curves. [5]

Markscheme

METHOD 1

substitutes their $\frac{dy}{dx}$ and the given $\frac{d^2y}{dx^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left|-\frac{r^2}{y^3}\right|}{\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left|-\frac{r^2}{y^3}\right| = \frac{r^2}{y^3}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1 + \left(-\frac{x}{y}\right)^2\right)$ as $\frac{y^2+x^2}{y^2}$ ***M1***

denominator is $\left(\frac{r^2}{y^2}\right)^{\frac{3}{2}}$ $\left(= \frac{r^3}{y^3}\right)$ ***A1***

Note: Award the following ***A*** mark for showing that $k = \frac{1}{r}$.

$k = \frac{\frac{r^2}{y^3}}{\frac{r^3}{y^3}}$ $\left(k = \frac{r^2}{y^3} \times \frac{y^3}{r^3}\right) \Rightarrow k = \frac{1}{r}$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left|-\frac{r^2}{y^3}\right| = \frac{r^2}{y^3}$ is not considered.

METHOD 2

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left|-\frac{r^2}{(r^2-x^2)^{\frac{3}{2}}}\right|}{\left(1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left|-\frac{r^2}{(r^2-x^2)^{\frac{3}{2}}}\right| = \frac{r^2}{(r^2-x^2)^{\frac{3}{2}}}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)$ as $\frac{r^2}{r^2-x^2}$ ***M1***

denominator is $\left(\frac{r^2}{r^2-x^2}\right)^{\frac{3}{2}}$ ***A1***

Note: Award the following ***A*** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{r^2}{(r^2-x^2)^{\frac{3}{2}}}}{\frac{r^3}{(r^2-x^2)^{\frac{3}{2}}}} \left(k = \frac{r^2}{(r^2-x^2)^{\frac{3}{2}}} \times \frac{(r^2-x^2)^{\frac{3}{2}}}{r^3}\right) \Rightarrow k = \frac{1}{r}$$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left|-\frac{r^2}{(r^2-x^2)^{\frac{3}{2}}}\right| = \frac{r^2}{(r^2-x^2)^{\frac{3}{2}}}$ is not considered.

METHOD 3

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left|-\frac{x^2+y^2}{y^3}\right|}{\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left|-\frac{x^2+y^2}{y^3}\right| = \frac{x^2+y^2}{y^3}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}$ as $\left(\frac{y^2+x^2}{y^2}\right)^{\frac{3}{2}}$ ***M1***

Note: Award the following ***A1A1*** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{x^2+y^2}{y^3}}{\left(\frac{x^2+y^2}{y^2}\right)^{\frac{3}{2}}} \left(\frac{\frac{x^2+y^2}{y^3}}{\frac{(x^2+y^2)^{\frac{3}{2}}}{y^3}} \right)$$

$= \frac{1}{(x^2+y^2)^{\frac{1}{2}}}$ ***A1***

$k = \frac{1}{(r^2)^{\frac{1}{2}}} \Rightarrow k = \frac{1}{r}$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left|-\frac{x^2+y^2}{y^3}\right| = \frac{x^2+y^2}{y^3}$ is not considered.

[5 marks]

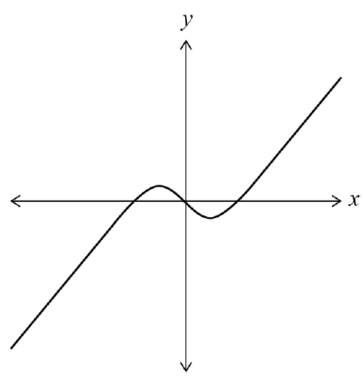
15. [Maximum mark: 28] 25M.3.AHL.TZ2.1

The following question explores features of a family of curves. The family is then linked to a homogeneous differential equation.

Consider the curve given by $y = \frac{x(x^2-16)}{x^2-16}$.

(a.i) Sketch the curve of y for $-10 \leq x \leq 10$. [1]

Markscheme



general shape of curve passing through $(0,\ 0)$ ***A1***

[1 mark]

(a.ii) State the coordinates of the points where the curve crosses the x-axis. [1]

Markscheme

$(-4,\ 0),\ (0,\ 0),\ (4,\ 0)$ ***A1***

Note: Award ***A0*** if additional points are given.

[1 mark]

(a.iii) State the coordinates of the local maximum point and the coordinates of the local minimum point. [2]

Markscheme

Max $(-1.94,\ 1.20)\,(-1.94347\ldots,\ 1.20113\ldots)$ ***A1***

Min $(1.94,\ -1.20)\,(1.94347\ldots,\ -1.20113\ldots)$ ***A1***

[2 marks]

(b) State whether the function $f(x)=\frac{x(x^2-16)}{x^2+16}$ is odd, even or neither. Justify your answer. [2]

Markscheme

EITHER

odd ***A1***

since $f(-x)=\frac{(-x)\left((-x)^2-16\right)}{(-x)^2+16}=\frac{-x(x^2-16)}{x^2+16}=-f(x)$ (or equivalent) ***R1***

OR

odd ***A1***

since graph of f has rotational symmetry order 2 (of 180 degrees) about $(0,\ 0)$ ***R1***

OR

odd ***A1***

since graph has reflection symmetry over the x-axis followed by reflection symmetry over the y-axis (or vice versa) ***R1***

Note: Do not award ***A1R0.***

[2 marks]

Now consider the general curve given by $y=\frac{x(x^2-A)}{x^2+A}$, where A is a positive constant and $x\in\mathbb{R}$.

(c) Given $f(x)=\frac{x(x^2-A)}{x^2+A}$, prove that $f\prime\left(\sqrt{A}\right)$ is independent of A. [4]

Markscheme

attempt to use quotient rule or product rule ***(M1)***

$f\prime(x) = \frac{(x^2+A)(3x^2-A)-2x(x^3-Ax)}{(x^2+A)^2}$ $\left(f\prime(x) = \frac{2x^2+(x^2-A)}{x^2+A} - \frac{2x^2+(x^2-A)}{(x^2+A)^2}\right)$ (or equivalent) ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

Note: If product rule used, award ***A1*** for each correct term.

Note: Award ***(M1)*** for an attempt to use quotient rule or product rule with $A = 16$ or any other constant. Award no further marks

$f\prime\left(\sqrt{A}\right) \left(= \frac{(2A)(2A)-0}{(2A)^2}\right) = 1$ ***A1***

hence independent of A ***AG***

[4 marks]

(d.i) Show that $x - \frac{2Ax}{x^2+A} \equiv \frac{x(x^2-A)}{x^2+A}$. [2]

Markscheme

METHOD 1

$x - \frac{2Ax}{x^2+A} \equiv \frac{x(x^2+A)-2Ax}{x^2+A}$ $\left(\equiv \frac{x^3+Ax-2Ax}{x^2+A}\right)$ ***M1***

$\equiv \frac{x^3-Ax}{x^2+A}$ $\left(\equiv \frac{x(x^2+A-2A)}{x^2+A}\right)$ ***A1***

$x - \frac{2Ax}{x^2+A} \equiv \frac{x(x^2-A)}{x^2+A}$ ***AG***

Note: Award no marks for numerical approaches.

METHOD 2

EITHER

attempt to do long division on $\frac{x(x^2-A)}{x^2+A}$ or $\frac{x^3-Ax}{x^2+A}$ ***M1***

quotient of x and remainder of $-2Ax$ ***A1***

$x - \frac{2Ax}{x^2+A} \equiv \frac{x(x^2-A)}{x^2+A}$ ***AG***

OR

$\frac{x^3}{x^2+A} - \frac{Ax}{x^2+A} \equiv \frac{x^3+Ax}{x^2+A} - \frac{Ax+Ax}{x^2+A}$ ***M1***

$\equiv \frac{x(x^2+A)}{(x^2+A)} - \frac{2Ax}{(x^2+A)}$ ***A1***

$\equiv x - \frac{2Ax}{x^2+A} \equiv \frac{x\left(x^2-A\right)}{x^2+A}$ ***AG***

Note: Award no marks for numerical approaches.

[2 marks]

(d.ii) Hence, determine the equation of the oblique asymptote to the curve. [1]

Markscheme

hence oblique asymptote is $y = x$ (since $\frac{2Ax}{x^2+A} \to 0$ as $x \to \pm\infty$) ***A1***

Note: Award ***A0*** if not given as an equation.

[1 mark]

(d.iii) Write down the coordinates of a point on the curve where the oblique asymptote is parallel to the tangent to the curve at that point. [1]

Markscheme

$\left(\sqrt{A},\ 0\right)$ **OR** $\left(-\sqrt{A},\ 0\right)$ ***A1***

[1 mark]

Now consider the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} = x(x+y) - y^2$, where $x \neq 0,\ y \neq \pm x$.

Using the substitution $y = vx$, the differential equation can be written as $x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - v^2$.

(e) Using partial fractions, show that $\int \frac{1}{1-v^2}\ \mathrm{d}\,v = \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$, where A is a positive constant. [5]

Markscheme

Note: Award no marks for attempts at integration that do not involve partial fractions.

attempt to write $\frac{1}{1-v^2}$ in the form $\frac{B}{1+v} + \frac{C}{1-v}$ ***(M1)***

$B = \frac{1}{2}$ and $C = \frac{1}{2}$ ***A1***

$\int \frac{1}{1-v^2}\ \mathrm{d}\,v = \int \frac{\frac{1}{2}}{1+v} + \frac{\frac{1}{2}}{1-v}\ \mathrm{d}\,v$

attempt to integrate their partial fractions ***(M1)***

EITHER

$\frac{1}{2}\ln|1+v| - \frac{1}{2}\ln|1-v| + \frac{1}{2}\ln A$ ***A1A1***

Note: Award ***A1*** for first two terms. The moduli must be present for this mark. Award ***A1*** for $\frac{1}{2}\ln\ A$.

OR

$\frac{1}{2}\ln|1+v| - \frac{1}{2}\ln|1-v|(+c)$ ***A1***

$= \frac{1}{2}\ln\left|\frac{e^{2c}(1+v)}{1-v}\right|$ where $e^{2c} = A$ ***A1***

Note: The moduli must be present for each of these marks.

THEN

$= \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$ ***AG***

[5 marks]

(f) Hence, show that a solution to the original differential equation may be expressed in the form $x^2 = \left|\frac{A(x+y)}{x-y}\right|$, where A is a positive constant. [5]

Markscheme

attempt to separate variables and integrate ***(M1)***

$\int \frac{dx}{x} = \int \frac{dv}{1-v^2}$

$\ln|x| = \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$ ***A1***

attempt to use power rule of logs ***(M1)***

$\ln|x| = \ln\left|\frac{A(1+v)}{1-v}\right|^{\frac{1}{2}}$ **OR** $\ln|x|^2 = \ln\left|\frac{A(1+v)}{1-v}\right|$

$|x| = \left|\frac{A(1+v)}{1-v}\right|^{\frac{1}{2}}$ **OR** $x^2 = \left|\frac{A(1+v)}{1-v}\right|$ ***A1***

$x^2 = \left|\frac{A\left(1+\frac{y}{x}\right)}{1-\frac{y}{x}}\right|$ ***A1***

$x^2 = \left|\frac{A(x+y)}{x-y}\right|$ ***AG***

Note: Award at most ***(M1)A1(M1)A1A0*** for incorrect use or omission of moduli.

Note: The substitution of $\frac{y}{x}$ may be seen earlier.

[5 marks]

Now consider only the case where $\frac{A(x+y)}{x-y} > 0$.

(g) Show that a solution to the original differential equation is $y=\frac{x\left(x^2-A\right)}{x^2+A}$. [4]

Markscheme

$x^2=\frac{A(x+y)}{x-y}$ ***(A1)***

attempt to clear denominator ***M1***

$x^2(x-y)=A(x+y)$

$x^3-x^2y=Ax+Ay$

attempt to isolate y ***M1***

$x^3-Ax=x^2y+Ay$

$x\left(x^2-A\right)=y\left(x^2+A\right)\ \left(x^3-Ax=y\left(x^2+A\right)\right)$ ***A1***

$y=\frac{x\left(x^2-A\right)}{x^2+A}$ ***AG***

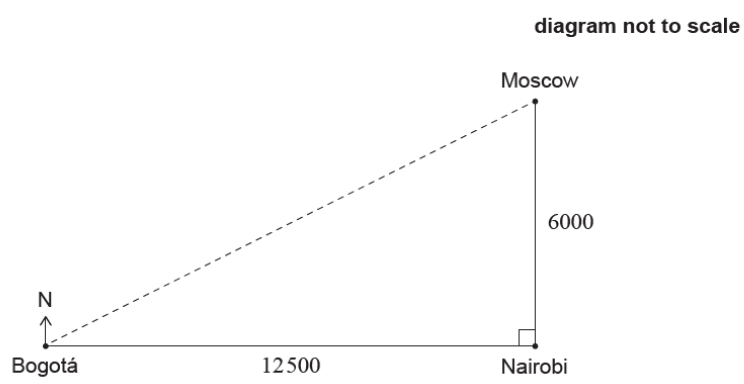
Note: The y must be factored out for the final ***A1***.

[4 marks]

16. [Maximum mark: 27] 25M.3.AHL.TZ2.2

The following question compares the distance and direction between cities on a flat surface to the distance and direction between cities on a sphere.

Consider a model where the cities of Bogotá, Moscow, and Nairobi lie on a flat surface. In this model, Nairobi is $6000\ \mathrm{km}$ due south of Moscow and Bogotá is $12\,500\ \mathrm{km}$ due west of Nairobi, as shown in the following diagram.



(a.i) Find the distance from Bogotá to Moscow. [2]

Markscheme

attempt to use Pythagoras' theorem ***(M1)***

$\sqrt{12\,500^2 + 6000^2}$

$= 13\,900$ (km) $(13865.4\ldots)$ ***A1***

[2 marks]

(a.ii) Find the bearing of Moscow from Bogotá. Give your answer in degrees. [3]

Markscheme

attempt to use an appropriate inverse trig ratio ***(M1)***

e.g. $\arctan\left(\frac{12\,500}{6000}\right)$ **OR** $\arctan\left(\frac{6000}{12\,500}\right)$

identifying the correct angle ***(M1)***

$= (0)64.4^\circ$ **OR** 064.4° $(64.3589\ldots)$ ***A1***

Note: The two method marks can be awarded independently.

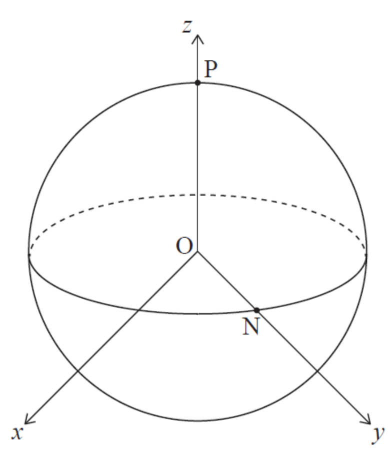
The first ***(M1)*** can come from any geometry leading to an inverse trigonometric ratio.

The second ***(M1)*** requires recognition of which angle is the bearing, possibly on a diagram. 64.4 appearing in intermediate working is not sufficient.

[3 marks]

In reality, these three cities lie on the curved surface of the Earth which will change the distances and directions found in part (a).

Now consider a curved model using a coordinate system $(x,\ y,\ z)$ with its origin, O, at the centre of the Earth. The units of this system are thousands of kilometres and the Earth is modelled as a sphere with radius $6000\ \mathrm{km}$. The North Pole, P, lies on the z-axis, and Nairobi, N, is modelled as being on the equator and lying on the y-axis.



P has position vector $\overrightarrow{OP} = \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ and N has position vector $\overrightarrow{ON} = \boldsymbol{n} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$.

(b.i) Use the scalar product to find the angle between $\boldsymbol{p}$ and $\boldsymbol{n}$. [2]

Markscheme

$\boldsymbol{p}.\boldsymbol{n} = 0$ ***M1***

so the angle is 90° **OR** $\frac{\pi}{2}$ rads ***A1***

Note: Award ***M0A0*** for an answer of 90 degrees without any mention that the scalar product is zero.

[2 marks]

(b.ii) Show that the distance between P and N along the arc from P to N is 3000π km. [2]

Markscheme

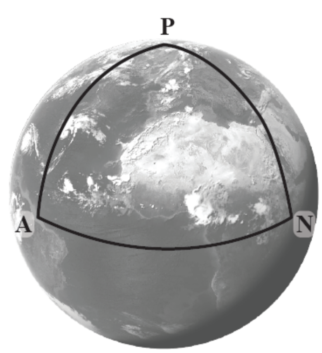
attempt to use arc length formula $s = r\theta$ **OR** $\frac{90^\circ}{360}$ of circumference ***(M1)***

$= 6000\left(\frac{\pi}{2}\right)$ **OR** $\frac{1}{4} \times 2\pi \times 6\,000$ ***A1***

$= 3\pi$ thousand km **OR** 3000π km ***AG***

[2 marks]

Point A, which is also on the equator, has position vector $\boldsymbol{a} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ as shown in the following diagram.



[Google Maps/Google Earth. Data from Data SIO, NOAA, U.S. Navy, NGA, GEBCOLandsat / Copernicus, IBCAO, U.S. Geological Survey, PGC/NASA. Imagery from 14/12/2015. Image available at: https://earth.google.com/web/@12.01529518,-18.56070747,-158.39383184a,23597813.93249989d,30.00008083y,359.99981502h,0t,0r/ data=CgRCAggBOgMKATBCAggASg0I____________ARAA.]

P, N and A, and the arcs connecting them, form a spherical triangle.

The angle at vertex A is defined as the angle between the vectors $\boldsymbol{a} \times \boldsymbol{p}$ and $\boldsymbol{a} \times \boldsymbol{n}$.

(c.i) Find the vector $\boldsymbol{a} \times \boldsymbol{p}$. [2]

Markscheme

attempt to find cross product (e.g. correct except for sign error) *(M1)*

$\boldsymbol{a} \times \boldsymbol{p} = \begin{pmatrix} 0 \\ -36 \\ 0 \end{pmatrix}$ ***A1***

[2 marks]

(c.ii) Show that the angle at vertex A in the spherical triangle is 90°. [3]

Markscheme

METHOD 1

$\boldsymbol{a} \times \boldsymbol{n} = \begin{pmatrix} 0 \\ 0 \\ 36 \end{pmatrix}$ ***A1***

choosing to use scalar product for their $\boldsymbol{a} \times \boldsymbol{p}$ and $\boldsymbol{a} \times \boldsymbol{n}$ ***M1***

$(\boldsymbol{a} \times \boldsymbol{p}).\,(\boldsymbol{a} \times \boldsymbol{n}) = 0$ ***A1***

Note: The final ***A1*** can be awarded for seeing the dot product equals zero without incorrect working. Follow through from (i) is permitted and if they explicitly state that they are using $\boldsymbol{a} \times \boldsymbol{n}$ then allow internal follow through.

so angle is 90° ***AG***

METHOD 2

$\boldsymbol{a} \times \boldsymbol{n} = \begin{pmatrix} 0 \\ 0 \\ 36 \end{pmatrix}$ ***A1***

attempt to use cross product for their $\boldsymbol{a} \times \boldsymbol{p}$ and $\boldsymbol{a} \times \boldsymbol{n}$. This must include both an attempt to find the cross product and knowledge that $\sin\theta = \frac{|x \times y|}{|x||y|}$ ***M1***

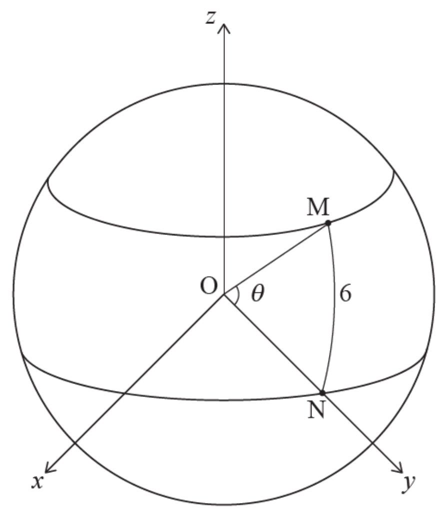
$(\boldsymbol{a} \times \boldsymbol{n}) \times (\boldsymbol{a} \times \boldsymbol{p}) = \begin{pmatrix} 1296 \\ 0 \\ 0 \end{pmatrix}$ **OR** $(\boldsymbol{a} \times \boldsymbol{p}) \times (\boldsymbol{a} \times \boldsymbol{n}) = \begin{pmatrix} -1296 \\ 0 \\ 0 \end{pmatrix}$

$\sin\theta = \frac{1296}{36 \times 36} = 1$ ***A1***

so angle is 90° ***AG***

[3 marks]

Moscow, M, has position vector $\overrightarrow{\mathrm{OM}} = \boldsymbol{m} = \begin{pmatrix} 0 \\ 6\cos\theta \\ 6\sin\theta \end{pmatrix}$, as shown in the following diagram.



The shortest distance between two points on the sphere lies along an arc of a circle on the sphere with centre O. In this model the shortest distance from Moscow to Nairobi is $6000\ \mathrm{km}$.

(d) Show that $\theta = 57.3^\circ$, correct to three significant figures. [2]

Markscheme

attempt to use $\frac{\theta}{360} \times 2\pi r = 6$ **OR** $s = r\theta$ ***(M1)***

$\frac{\theta}{360} \times 2\pi \times 6 = 6$ **OR** $6 = 6\theta \Rightarrow \theta = 1$ radian

$\theta = 57.2957\ldots^\circ$ ***A1***

Note: The fourth significant figure must be seen somewhere to award the ***A1***.

$\theta = 57.3^\circ$ to 3 significant figures ***AG***

[2 marks]

Bogotá, B, is west of Nairobi and has position vector $\overrightarrow{\mathrm{OB}} = \boldsymbol{b} = \begin{pmatrix} 6\sin 120^\circ \\ 6\cos 120^\circ \\ 0 \end{pmatrix}$.

(e) Find the shortest distance from Bogotá to Moscow on the sphere. [5]

Markscheme

METHOD 1

EITHER (Scalar product)

attempt to use scalar product to find $\mathrm{B\hat{O}M} = \arccos\left(\frac{\boldsymbol{b}.\boldsymbol{m}}{|\boldsymbol{b}||\boldsymbol{m}|}\right)$ ***M1***

Note: This may be written as $\cos\left(\mathrm{B\hat{O}M}\right) = \frac{\boldsymbol{b}.\boldsymbol{m}}{|\boldsymbol{b}||\boldsymbol{m}|}$.

$= \text{arc } \cos\left(\frac{(6\cos 120^\circ)(6\cos 57.3^\circ)}{6^2}\right)$ ***(A1)***

Note: This ***(A1)*** can be awarded for $\cos\theta = \frac{6\cos 120^\circ \times 6\cos 57.3^\circ}{6^2}$.

$= 106^\circ \ (= 105.673\ldots^\circ)$ **OR** 1.84 radians $(1.84434\ldots)$ ***(A1)***

OR (Cosine Rule)

$|\text{BM}| = \sqrt{(6\sin 120 - 0)^2 + (6\cos 120 - 6\sin 57.3)^2 + (0 - 6\sin 57.3)^2}$

$= 9.56 \ (9.56299\ldots)$ **OR** 9560 km ***(A1)***

attempt to use the cosine rule to find angle BOM ***(M1)***

$|\text{BM}|^2 = 6^2 + 6^2 - 2 \times 6 \times 6 \times \cos \text{ BOM}$

$\text{BOM} = 106^\circ \ (= 105.673\ldots^\circ)$ **OR** 1.84 radians $(1.84434\ldots)$ ***(A1)***

THEN

attempt to use $\frac{\theta}{360} \times 2\pi r$ **OR** $s = r\theta$ ***(M1)***

$= \frac{105.671^\circ}{360} \times 2\pi \times 6$

$= 11.1$ $(11.0660\ldots$ thousand km) ***A1***

Note: Accept an answer of $11\,100$ **OR** $11\,066.0\ldots$ without units (i.e. "km" is implicit).

METHOD 2 (Cross product)

attempt to use cross product to find $\widehat{\text{BOM}} = \text{arc } \sin \frac{|\boldsymbol{b}\times\boldsymbol{m}|}{|\boldsymbol{b}||\boldsymbol{m}|}$ ***(M1)***

Note: This may be written as $\sin \widehat{\text{BOM}} = \frac{|\boldsymbol{b}\times\boldsymbol{m}|}{|\boldsymbol{b}||\boldsymbol{m}|}$.

$= \frac{\left|\begin{pmatrix} 36\cos 120\sin\theta \\ -36\sin 120\sin\theta \\ 36\sin 120\cos\theta \end{pmatrix}\right|}{6\times 6}$

$= \frac{\sqrt{\sin^2\theta + 3}}{2} = 0.962818\ldots$ ***(A1)***

$\text{B}\hat{\text{O}}\text{M} = 180 - \arcsin 0.962818\ldots$

$= 106^\circ \ (= 105.673\ldots,\ 1.844346 \text{ rads})$ ***(A1)***

attempt to use $\frac{\theta}{360} \times 2\pi r$ **OR** $s = r\theta$ ***(M1)***

$= \frac{105.671^\circ}{360} \times 2\pi \times 6$

$= 11.1$ ($11.0660\ldots$ thousand km) ***A1***

Note: Award at most ***(M1)(A1)(A0)(M1)A0*** for an answer of 7.78 thousand km from finding the principal root of the arcsin.

[5 marks]

The bearing from B to M is defined as the angle at vertex B in the spherical triangle containing $\text{B},\ \text{M}$ and P. It is given that $\boldsymbol{b} \times \boldsymbol{p} = \begin{pmatrix} 36 \cos 120^\circ \\ -36 \sin 120^\circ \\ 0 \end{pmatrix}$.

(f) Using the method from part (c), find the bearing from Bogotá to Moscow. [6]

Markscheme

attempt to calculate $\boldsymbol{b} \times \boldsymbol{m}$ ***(M1)***

$$\boldsymbol{b} \times \boldsymbol{m} = \begin{pmatrix} 36 \cos 120^\circ \sin 57.3^\circ \\ -36 \sin 120^\circ \sin 57.3^\circ \\ 36 \sin 120^\circ \cos 57.3^\circ \end{pmatrix} \left(= \begin{pmatrix} -15.1464\ldots \\ -26.2344\ldots \\ 16.8449\ldots \end{pmatrix} \right)$$ ***(A1)***

attempt to use scalar product to find angle between vectors ***(M1)***

$(\cos \theta =) \left(\frac{(\boldsymbol{b} \times \boldsymbol{m}).(\boldsymbol{b} \times \boldsymbol{p})}{|\boldsymbol{b} \times \boldsymbol{m}||\boldsymbol{b} \times \boldsymbol{p}|} \right)$

$(\cos \theta =) \left(\frac{1090.54\ldots}{34.6614\ldots \times 36} \right)$ ***(A1)(A1)***

Note: Award ***(A1)*** for numerator, ***(A1)*** for denominator.

bearing is $(0)29.1^\circ \ (29.0770\ldots^\circ)$ ***A1***

Note: Do not penalise absence of degree symbol.
Award full marks for answer in radians $= 0.507$.
Award full marks for 029°.
Special Case: If a candidate calculates $\boldsymbol{m} \times \boldsymbol{b}$ instead of $\boldsymbol{b} \times \boldsymbol{m}$ then you can award at most ***(M1)(A0)(M1)(A1)(A1)A0***.

[6 marks]

17. [Maximum mark: 27] 25M.3.AHL.TZ3.1

This question asks you to explore self-composite linear functions, of the form $f(\mathrm{x}) = mx + c$, for varying values of m.

A function composed with itself is called a self-composite function.

For a function f, the function composition with itself is given by $(f \circ f)(x) = f(f(x))$.

Let f^n denote the $n\,\mathrm{th}$ composition of f with itself where $f^n(x) = \underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}}(x)$.

Hence, for example, $f^2(x) = (f \circ f)(x)$ and $f^3(x) = (f \circ f \circ f)(x) = f(f(f(x)))$.

Consider the linear function $f(\mathrm{x}) = mx + c$, where $x \in \mathbb{R}$ and $m,\ c \in \mathbb{R}$.

(a) Show that

(a.i) $f^2(x) = m^2x + c(1+m)$; [3]

Markscheme

attempts to find an expression for $f^2(x)$ ***(M1)***

Note: For example, award ***(M1)*** for $f(mx+c)$ seen.

$= m(mx+c) + c$ ***A1***

$= m^2x + cm + c$ ***A1***

$f^2(x) = m^2x + c(1+m)$ ***AG***

[3 marks]

(a.ii) $f^3(x) = m^3x + c\left(1 + m + m^2\right)$. [2]

Markscheme

attempts to find an expression for $f^3(x)$ ***(M1)***

Note: For example, award ***(M1)*** for $f^2(mx+c)$ or $f\left(m^2x + c(1+m)\right)$ seen.

EITHER

$f^3(x) = m^2(mx+c) + c(1+m)\ \left(= m^3x + m^2c + c + cm\right)$ ***A1***

OR

$f^3(x) = m\left(m^2x + c(1+m)\right) + c\ \left(= m^3x + mc + m^2c + c\right)$ ***A1***

THEN

$f^3(x) = m^3x + c(1 + m + m^2)$ ***AG***

[2 marks]

(b.i) Write down an expression for $f^4(x)$. [1]

Markscheme

$f^4(x) = m^4x + c(1 + m + m^2 + m^3)$ ***A1***

Note: Award ***A1*** for equivalent forms, for example:

$f^4(x) = m^4x + c\sum_{1=0}^{3} m^i$

$f^4(x) = m^3(mx + c) + c(1 + m + m^2)$

$f^4(x) = m(m^3x + m^2c + mc + c) + c$

[1 mark]

(b.ii) Suggest a similar expression for $f^n(x),\ n \in \mathbb{Z}^+$. [2]

Markscheme

$f^n(x) = m^nx + c(1 + m + (m^2+)\ldots+m^{n-1})$ ***A1A1***

Note: Award ***A1*** for m^nx and ***A1*** for $c(1 + m + m^2+\ldots+m^{n-1})$ or equivalent, for example: $c\sum_{i=0}^{n-1} m^1$ **OR** $c\left(\frac{1-m^n}{1-m}\right)$
For alternative forms, award ***A1*** for the correct term in x and ***A1*** for the correct constant term.

[2 marks]

(b.iii) By using your expression from part (b)(ii), or otherwise, find an expression in terms of n for $f^n(x)$ when $m = 1$. [3]

Markscheme

METHOD 1

attempts to substitute $m = 1$ into their $f^n(x)$ ***(M1)***

$f^n(x) = x + c(1 + 1 + 1+\ldots+1)$ ***(A1)***

$f^n(x) = x + nc$ ***A1***

METHOD 2

$f(x) = x + c$ ***(A1)***

attempts to find $f^2(x)$ with their $f(x)$ and $m = 1$ ***(M1)***

$f^2(x) = f(x + c) = x + 2c$

$\Rightarrow f^n(x) = x + nc$ ***A1***

[3 marks]

(c) For $m \neq 1$, use mathematical induction to prove that

$f^n(x) = m^n x + c\left(\frac{1-m^n}{1-m}\right)$ for $n \in \mathbb{Z}^+$. [8]

Markscheme

consider $n = 1$

$f^1(x) = m^1 x + c\left(\frac{1-m^1}{1-m}\right) \Rightarrow f(x) = mx + c,$ ***R1***

hence true for $n = 1$

Note: The substitution $n = 1$ must be explicit for the ***R1*** to be awarded.
Award ***R0*** for only considering $n = 2$.
Subsequent marks after this ***R1*** mark are independent of this mark and can be awarded.

assume true for $n = k$ ($p(k)$ is true) ***M1***

$f^k(x) = m^k x + c\left(\frac{1-m^k}{1-m}\right)$ (for some $k \in \mathbb{Z}^+$)

Note: The assumption of truth must be apparent. Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true" or "assume $n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

EITHER

Attempting to find $f\left(f^k(x)\right)$ ***(M1)***

$f^{k+1}(x) = f\left(f^k(x)\right)$

$f\left(m^k x + c\left(\frac{1-m^k}{1-m}\right)\right) = m\left(m^k x + c\left(\frac{1-m^k}{1-m}\right)\right) + c$ ***A1***

$= \left(m^{k+1} x + cm\left(\frac{1-m^k}{1-m}\right)\right) + c$

OR

Attempting to find $f^k(f(x))$ ***(M1)***

$f^{k+1}(x) = f^k(f(x))$

$f^k(mx + c) = m^k(mx + c) + c\left(\frac{1-m^k}{1-m}\right)$ ***A1***

$= \left(m^{k+1} x + cm^k + c\left(\frac{1-m^k}{1-m}\right)\right)$

THEN

EITHER

attempts to put c over a common denominator *(M1)*

$m^{k+1}x + cm\left(\frac{1-m^k}{1-m}\right) + \frac{c(1-m)}{1-m}$

$= m^{k+1}x + c\left(\frac{m(1-m^k)(1-m)}{1-m}\right)$ (or equivalent) *A1*

OR

attempts to put cm^k over a common denominator *(M1)*

$m^{k+1}x + \frac{cm^k(1-m)}{1-m} + c\left(\frac{1-m^k}{1-m}\right)$

$m^{k+1}x + c\left(\frac{m^k(1-m)+(1-m^k)}{1-m}\right)$ (or equivalent) *A1*

OR

recognizes that $\frac{1-m^k}{1-m}$ can be expressed as a geometric series *(M1)*

$m^{k+1}x + cm\left(1 + m + m^2 + \ldots + m^{k-1}\right) + c$

$= m^{k+1}x + c\left(1 + m + m^2 + m^3 + \ldots + m^k\right)$ *A1*

THEN

$= m^{k+1}x + c\left(\frac{1-m^{k+1}}{1-m}\right)$ *A1*

since true for $n = 1$ and true for $n = k+1$ if true for $n = k$, hence true for all $n(\in \mathbb{Z}^+)$ *R1*

Note: To obtain the final ***R1,*** four of the previous seven marks must have been awarded.
The statement 'true for $n = 1$' may be seen anywhere in the proof.

[8 marks]

Consider $f^n(x) = m^n x + c\left(\frac{1-m^n}{1-m}\right)$ where $-1 < m < 1$.

(d) As $n \to \infty$, the family of graphs $y = f^n(x)$ approaches the graph of a straight line, L.

Determine the equation of L, giving your answer in terms of c and m. [4]

Markscheme

attempts to apply $m^n \to 0$ (as $n \to \infty$) to at least one term in expression *(M1)*

Note: Award *(M1)* for an equivalent statement/conclusion in words.

$m^n x \to 0$ $\left(c\left(\frac{1-m^n}{1-m}\right) \to\right) \frac{c}{1-m}$ (as $n \to \infty$) *(A1)A1*

Note: Award ***(A1)*** for the correct term in x and ***A1*** for the correct constant term.

$y=\frac{c}{1-m}$ ***A1***

[4 marks]

Consider $f^n(x)=m^n x+c\left(\frac{1-m^n}{1-m}\right)$ where $m=-1$.

(e.i) Show that $f^n(x)=-x+c$ when n is odd. [2]

Markscheme

$f^n(x)=(-1)^n x+c\left(\frac{1-(-1)^n}{1-(-1)}\right)$ ***A1***

Note: Award ***A1*** here if the above is seen in part (e)(ii).
Award ***A0*** for a missing bracket, for example: $f^n(x)=-1^n x+c\left(\frac{-1(-1)^n}{-1(-1)}\right)$.

$(-1)^n=-1$ (when n is odd) ***R1***

$\left(f^n(x)=-x+c\left(\frac{1-(-1)}{1-(-1)}\right)\right)$

Note: The ***A1*** and ***R1*** mark can be awarded in either order. ***A0R1*** is possible.
Award ***R0*** for arguments based on specific numerical examples.

$f^n(x)=-x+c$ ***AG***

[2 marks]

(e.ii) Find an expression for $f^n(x)$ when n is even. [2]

Markscheme

evidence that an even n has been considered ***(M1)***

for example, $(-1)^2=1$

$f^n(x)=x$ ***A1***

Note: Award ***(M1)A1*** for answer of x (need not be an equation).

[2 marks]

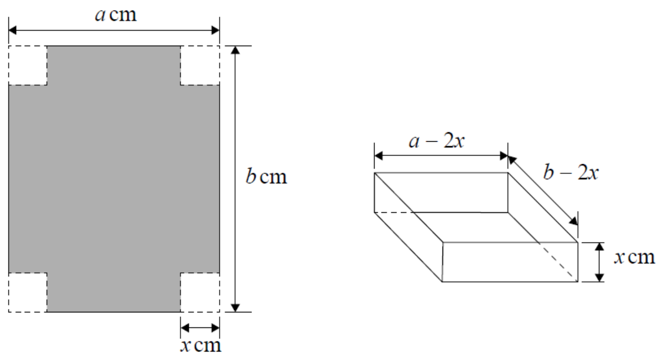
18. [Maximum mark: 28] 25M.3.AHL.TZ3.2

This question asks you to consider open-topped boxes formed by cutting square corners from an $a \times b$ sheet of cardboard and folding the sides. In particular, you are asked to investigate conditions for which the maximum volume of a box is an integer.

A rectangular sheet of cardboard measures $a\,\text{cm}$ by $b\,\text{cm}$ where $a < b$.

A square of side length $x\,\text{cm}$ is cut from each corner and the resulting sheet of cardboard is then folded to form an open-topped box with a rectangular base.

This is shown in the following diagrams.



The volume, $V\,\text{cm}^3$, of a box is given by $V = x(a-2x)(b-2x)$ where $0 \leq x \leq \frac{a}{2}$.

(a) Write down the value of V when

(a.i) $x = 0$; [1]

Markscheme

0 ***A1***

[1 mark]

(a.ii) $x = \frac{a}{2}$. [1]

Markscheme

0 ***A1***

[1 mark]

(b) Show that $V = 4x^3 - 2(a+b)x^2 + abx$. [2]

Markscheme

attempts to expand a product of two of the side lengths ***(M1)***

EITHER

$V = x\left(4x^2 - 2(a+b)x + ab\right)$ $\left(V = x\left(4x^2 - 2ax - 2bx + ab\right)\right)$ ***A1***

OR

$V = \left(ax - 2x^2\right)(b - 2x)$

$= 4x^3 - 2ax^2 - 2bx^2 + abx$ ***A1***

OR

$V = (a - 2x)\left(bx - 2x^2\right)$

$= 4x^3 - 2ax^2 - 2bx^2 + abx$ ***A1***

THEN

$V = 4x^3 - 2(a + b)x^2 + abx$ ***AG***

[2 marks]

(c) Show that the only solutions of $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ are $x = \frac{(a+b)\pm\sqrt{a^2-ab+b^2}}{6}$. [6]

Markscheme

attempts to find $\frac{\mathrm{d}V}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 4(a + b)x + ab$ ***A1***

Note: Award ***(M1)*** for obtaining the equivalent of two correct terms.

EITHER

attempts to solve $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ for x using the quadratic formula ***(M1)***

$x = \frac{-(-4(a+b))\pm\sqrt{(-4(a+b))^2-4(12)(ab)}}{24}$ ***(A1)***

Note: Award ***A0*** for incorrect expressions, for example:

$x = \frac{-(-4(a+b))\pm\sqrt{-4(a+b)^2-4(12)(ab)}}{24}$

attempts to expand brackets within the square root sign and/or to take a common factor of 4 ***(M1)***

$x = \frac{-(-4(a+b))\pm\sqrt{16a^2+16b^2+32ab-48ab}}{24}$ **OR** $x = \frac{4(a+b)\pm4\sqrt{a^2+2ab+b^2-3ab}}{24}$ ***A1***

OR

attempt to substitute either of the given x values into their $\frac{\mathrm{d}V}{\mathrm{d}x}$ ***M1***

correct manipulation with each solution to reduce each expression to zero ***A1A1***

$\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ is a quadratic equation and hence these are the only two solutions ***R1***

THEN

$x = \frac{(a+b) \pm \sqrt{a^2-ab+b^2}}{6}$ ***AG***

[6 marks]

In parts (d) and (e), the following three results are given.

$x = \frac{(a+b) + \sqrt{a^2-ab+b^2}}{6}$ lies outside the interval $0 \leq x \leq \frac{a}{2}$.

$x = \frac{(a+b) - \sqrt{a^2-ab+b^2}}{6}$ lies inside the interval $0 \leq x \leq \frac{a}{2}$.

$\sqrt{a^2 - ab + b^2} > 0$.

Let $x_m = \frac{(a+b) - \sqrt{a^2-ab+b^2}}{6}$.

(d) Use $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ to show that there is a local maximum of V at $x = x_m$. [5]

Markscheme

attempts to find their $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24x - 4(a+b)$ ***A1***

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} < 0$ ***R1***

Note: Award ***R1*** for equivalent statement in words.

substitutes $x = \frac{(a+b) - \sqrt{a^2-ab+b^2}}{6}$ into their $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ ***(M1)***

Note: Award ***(M1)*** for substituting $x = \frac{(a+b) \pm \sqrt{a^2-ab+b^2}}{6}$

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} x = 24\left(\frac{(a+b) - \sqrt{a^2-ab+b^2}}{6}\right) - 4(a+b)$

$= -4\sqrt{a^2 - ab + b^2}\ (< 0)$ ***A1***

so $x_m = \frac{(a+b) - \sqrt{a^2-ab+b^2}}{6}$ gives a local maximum of V ***AG***

[5 marks]

(e) Use your answers to parts (a) and (d) to explain why x_m will give the maximum volume, V_m, of the box. [1]

Markscheme

the maximum volume will occur at either the endpoints or at the local maximum. ***R1***

Note: Award ***R1*** for an answer which refers to the volume being 0 at both endpoints ($x = 0$ and $x = \frac{a}{2}$) and the volume of the box at $x_m > 0$

hence the maximum volume of the box occurs at x_m / the local maximum is a global maximum. ***AG***

[1 mark]

In parts (f) to (h), consider a method for generating sets of positive integer values for x_m and hence V_m.

Consider the case where $a = 2st - s^2,\ b = t^2 - s^2$, and $s,\ t \in \mathbb{Z}^+$.

(f) Given that $a < b$, show that $t > 2s$. [2]

Markscheme

$2st - s^2 < t^2 - s^2$ ***A1***

$t^2 > 2st$ and since $t \in \mathbb{Z}^+$ (dividing both sides by t) ***R1***

Note: Award ***R1*** for equivalent reasoning, $t > 0$ or t is positive

$\Rightarrow t > 2s$ ***AG***

[2 marks]

It is given that $\sqrt{a^2 - ab + b^2} = s^2 - st + t^2$. You are **not** required to show this result.

(g) By substituting for each of $a,\ b$ and $\sqrt{a^2 - ab + b^2}$ in terms of s and t, show that $x_m = \frac{s(t-s)}{2}$. [3]

Markscheme

attempts to substitute for each of $a,\ b$ and $\sqrt{a^2 - ab + b^2}$ in terms of s and t **(M1)**

$(x_m =)\ \frac{(2st-s^2)+(t^2-s^2)-(s^2-st+t^2)}{6}$ ***A1***

$= \frac{-3s^2+3st}{6}$ ***A1***

$x_m = \frac{s(t-s)}{2}$ ***AG***

[3 marks]

(h.i) Given that $a,\ b \in \mathbb{Z}^+$, use $V_m = x_m(a - 2x_m)(b - 2x_m)$ to explain why s being even implies that V_m is an integer.

[2]

Markscheme

METHOD 1

when s is even, x_m is an integer ***R1***

hence V_m is a product of integers $(a,\ b \in \mathbb{Z}^+)$ ***R1***

hence V_m is an integer ***AG***

METHOD 2

$V_m = \frac{s(t-s)}{2}\left(2st - s^2 - 2\left(\frac{s(t-s)}{2}\right)\right)\left(t^2 - s^2 - 2\left(\frac{s(t-s)}{2}\right)\right)$ (or equivalent) ***A1***

Note: Award ***A1*** for $V_m = \frac{s(t-s)}{2}\left(a - 2\left(\frac{s(t-s)}{2}\right)\right)\left(b - 2\left(\frac{s(t-s)}{2}\right)\right)$

when s is even, $\frac{s(t-s)}{2}$ is an integer **AND**

hence V_m is a product of integers $(a,\ b \in \mathbb{Z}^+)$ ***R1***

hence V_m is an integer ***AG***

[2 marks]

(h.ii) State another condition on s and a corresponding condition on t, that implies that both x_m and V_m are integers. [1]

Markscheme

when s is odd **AND** t is odd ($s(t-s)$ is even) ***A1***

x_m and V_m are both integers ***AG***

[1 mark]

(h.iii) Hence, or otherwise, find an integer value of x_m and the corresponding integer value of V_m. [4]

Markscheme

the following table shows some possible sets of values for $s,\ t,\ a, b,\ x_m$ and V_m

s	t	a	b	x_m	V_m
1	3	5	8	1	18
1	5	9	24	2	200
2	5	16	21	3	450
2	6	20	32	4	1152
2	7	24	45	5	2450
2	8	28	60	6	4608
3	7	33	40	6	3528
4	10	64	84	12	28800
…	…	…	…	…	…

attempts to use values of s and t to find

EITHER x_m **OR** a and b **OR** V_m ***(M1)***

a corresponding correct x_m value ***A1***

corresponding correct a and b values ***(A1)***

a corresponding correct V_m value ***A1***

Note: Award at most ***(M1)A0A0A0*** if the values of s and t do not satisfy the conditions: $s,\ t \in \mathbb{Z}^+,\ t > 2s$, and s is even / s and t are odd.
The three ***A*** marks can be awarded in any order.
The ***A1*** for x_m and the ***A1*** for a and b can be awarded independently.
Some candidates may determine in part (h) that $V_m = \frac{s^2t^2(t-s)^2}{2}$.

[4 marks]

19. [Maximum mark: 27] 24N.3.AHL.TZ0.1

This question asks you to investigate models for the population of trout in a lake.

Trout is a type of fish. At the start of a year, a lake is estimated to contain 6000 trout.

The owner of the lake estimates that the number of trout will increase by 10% per year.

At the end of each year, the owner proposes to remove 500 trout from the lake to prevent overpopulation.

Therefore, the relationship between T_n, the predicted number of trout at the start of year n, and T_{n+1}, the predicted number of trout at the start of year $n+1$, is given by

$T_{n+1} = 1.1T_n - 500$ and $T_1 = 6000$.

For example, the predicted number of trout at the start of the second year is given by

$T_2 = 1.1T_1 - 500$.

(a) Use this formula to verify that $T_2 = 6100$. [2]

Markscheme

$T_2 = 1.1(6000) - 500$ OR $T_2 = 6600 - 500$ ***A2***

$= 6100$ ***AG***

[2 marks]

(b.i) Verify that $T_3 = 6210$. [1]

Markscheme

$T_2 = 1.1(6100) - 500$ $(T_3 = 1.1(1.1(6000) - 500) - 500)$ OR $T_3 = 6710 - 500$ ***A1***

$= 6210$ ***AG***

[1 mark]

(b.ii) Find T_4. [2]

Markscheme

$T_4 = 1.1(6210) - 500$ OR $T_4 = 6831 - 500$ ***(A1)***

$= 6331$ ***A1***

[2 marks]

It is also known that $T_n = 6000(1.1)^{n-1} - \frac{500\left((1.1)^{n-1}-1\right)}{1.1-1}$.

(c.i) Show that $T_n = 1000(1.1)^{n-1} + 5000$. [2]

Markscheme

EITHER

attempts to eliminate the denominator ***(M1)***

$T_n = 6000(1.1)^{n-1} - 5000\left((1.1)^{n-1} - 1\right)$ $\left(= (6000 - 5000)(1.1)^{n-1} + 5000\right)$ ***A1***

Note: Accept correct equivalent forms, eg. $T_n = 6000(1.1)^{n-1} - 5000(1.1)^{n-1} + 5000$.

OR

attempts to express T_n with a common denominator of 0.1 ***(M1)***

$T_n = \frac{600(1.1)^{n-1} - 500\left((1.1)^{n-1}-1\right)}{0.1}$ $\left(= \frac{600(1.1)^{n-1} - 500(1.1)^{n-1} + 500}{0.1}\right)$ ***A1***

Note: Accept correct equivalent forms.

THEN

$T_n = 1000(1.1)^{n-1} + 5000$ ***AG***

[2 marks]

(c.ii) Hence, or otherwise, find T_6. Give your answer to the nearest whole number. [2]

Markscheme

EITHER

attempts to find T_6 using the explicit formula for T_n ***(M1)***

$T_6 = 1000(1.1)^{6-1} + 5000$ OR $T_6 = 6610.51$

OR

attempts to find T_6 using $T_{n+1} = 1.1T_n - 500$ ***(M1)***

$T_4 = 6331, T_5 = 6464.1, T_6 = 6610.51$

OR

attempts to find T_6 using a finance solver application ***(M1)***

$\mathrm{N} = 5, \mathrm{I} = 10\%, \mathrm{PV} = -6000$ and $\mathrm{PMT} = 500$

THEN

6611 ***A1***

Note: Accept 6610.

[2 marks]

After deciding that the trout population would increase too quickly, the lake owner proposes instead to remove 750 trout at the end of each year.

The relationship between D_n, the predicted number of trout at the start of year n, and D_{n+1}, the predicted number of trout at the start of year $n+1$, is now given by

$D_{n+1} = 1.1D_n - 750$ and $D_1 = 6000$.

It is also known that $D_n = -1500(1.1)^{n-1} + 7500$.

(d.i) Show that $D_{n+1} - D_n = -150(1.1)^{n-1}$. [3]

Markscheme

METHOD 1

attempts to use the explicit formula for D_n to find an expression for $D_{n+1} - D_n$ ***(M1)***

$= 1.1D_n - 750 - \left(-1500(1.1)^{n-1} + 7500\right)$

$= 1.1\left(-1500(1.1)^{n-1} + 7500\right) - 750 - \left(-1500(1.1)^{n-1} + 7500\right)$ ***A1***

EITHER

$= -1650(1.1)^{n-1} + 8250 - 750 + 1500(1.1)^{n-1} - 7500$ ***A1***

OR

$= 1500(1.1)^{n-1} - 1500(1.1)^{n}$

$= 1500(1.1)^{n-1}(1 - 1.1) \quad \left(= 1500(1.1)^{n-1}(-0.1)\right)$ ***A1***

THEN

$D_{n+1} - D_n = -150(1.1)^{n-1}$ ***AG***

METHOD 2

attempts to express $D_{n+1} - D_n$ in terms of D_n ***(M1)***

$D_{n+1} - D_n = 1.1D_n - 750 - D_n$

$= 0.1D_n - 750$ ***(A1)***

$= 0.1\left(-1500(1.1)^{n-1} + 7500\right) - 750 \quad \left(= -150(1.1)^{n-1} + 750 - 750\right)$ ***A1***

$D_{n+1} - D_n = -150(1.1)^{n-1}$ ***AG***

[3 marks]

(d.ii) Use the result in part (d)(i) to deduce that the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year. [1]

Markscheme

$D_{n+1} - D_n < 0$ (because $(1.1)^{n-1} > 0$ and so $-150(1.1)^{n-1} < 0$) ***R1***

Note: Award ***R1*** for responses such as:

'The difference is negative'.

$-150(1.1)^{n-1} < 0$

$D_n = D_{n+1} + 150(1.1)^{n-1}$

Accept responses such as:

$D_n - D_{n+1} > 0$ or 'the difference is positive' as long as $D_n - D_{n+1}$ is referred to.

Do not accept arguments based on specific numerical examples.

therefore $D_{n+1} < D_n \Rightarrow D_n > D_{n+1}$ ***AG***

(and hence the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year)

[1 mark]

(e) Determine the first year during which there will be no trout remaining in the lake. [4]

Markscheme

METHOD 1

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $D_n \leq 0$ ($D_n = 0$ OR $D_n < 0$) ***(M1)***

Note: Award ***(M1)*** for attempting to obtain $-1500(1.1)^{n-1} \leq -7500$ or $-1500(1.1)^{n-1} = -7500$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award ***(M1)***, for considering values of n either side of $D_n = 0$.

EITHER

$17.8863\ldots \quad \left(= \log_{1.1} 5 + 1, \ = \frac{\ln 5}{\ln 1.1} + 1\right)$ (critical value of n seen anywhere) ***(A2)***

OR

$D_{17} = 607.540\ldots \quad (607.540\ldots < 750)$ ***(A2)***

OR

$D_{18} = -81.7054\ldots \quad (< 0)$ **(A2)**

THEN

trout disappears from the lake during (at the end of) the 17th year ***A1***

Note: Award ***A1*** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

Note: Award as above for $D_{n+1} \leq 0$ ($D_{n+1} = 0$ OR $D_{n+1} < 0$).

METHOD 2

$S_n = \sum (D_{n+1} - D_n)$ is the total decrease in the number of trout by the end of year n or start of year $n + 1$

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $S_n \leq -6000$ ($S_n = -6000$ OR $S_n < -6000$) ***(M1)***

$\frac{-150((1.1)^n-1)}{1.1-1}$ $(= -1500((1.1)^n - 1)) \leq -6000$

Note: Award ***(M1)*** for attempting to obtain $-1500((1.1)^n - 1) \leq -6000$ or $-1500((1.1)^n - 1) = -6000$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award ***(M1)***, for considering values of n either side of $S_n = -6000$.

$n = 16.8863\ldots$ ***(A2)***

trout disappears from the lake during (at the end of) the 17th year ***A1***

Note: Award ***A1*** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

[4 marks]

The lake owner now considers a more general approach where d trout are removed at the end of each year.

Let C_n denote the predicted number of trout in the lake at the start of the $n\text{th}$ year where

$C_n = 6000(1.1)^{n-1} - 10d\left((1.1)^{n-1} - 1\right)$.

(f) Find the value of d such that the predicted number of trout at the start of each year is constant. [3]

Markscheme

METHOD 1

$C_n = (6000 - 10d)(1.1)^{n-1} + 10d$ ***(A1)***

EITHER

$6000 - 10d = 0$ (recognizes that C_n is independent of n) ***(M1)***

OR

$10d = 6000$ (sets $C_n = 6000$ and compares coefficients) ***(M1)***

THEN

$d = 600$ ***A1***

METHOD 2

EITHER

$C_{n+1} - C_n \ (= 0)$

$6000(1.1)^n - 10d((1.1)^n - 1) - \left(6000(1.1)^{n-1} - 10d\left((1.1)^{n-1} - 1\right)\right) \quad (= 0)$ ***(A1)***

OR

$0.1C_n - d \quad (= 0)$ (similar to part (d) (i) method 2)

$0.1\left(6000(1.1)^{n-1} - 10d\left((1.1)^{n-1} - 1\right)\right) - d \quad (= 0)$ ***(A1)***

THEN

attempts to expand to eliminate the linear terms in d and then attempts to factorize ***(M1)***

$600(1.1)^{n-1} - d(1.1)^{n-1} \quad (= 0) \quad \left((600 - d)(1.1)^{n-1} \quad (= 0)\right)$

$d = 600$ ***A1***

METHOD 3

considers $C_{n+1} = 1.1C_n - d$ and $C_1 = 6000$ ***(A1)***

EITHER

$6000 = 1.1 \times 6000 - d \quad (C_1 = C_2 = 6000)$ ***(A1)***

OR

systematically varies the value of d ***(M1)***

THEN

$d = 600$ ***A1***

METHOD 4

EITHER

considers a specific $C_n = C_{n+1}$, for example, $C_2 = C_3$ ***(A1)***

$6000(1.1) - 10d(1.1 - 1) = 6000(1.1)^2 - 10d\left((1.1)^2 - 1\right)$

attempts to solve their $C_n = C_{n+1}$ for d ***(M1)***

for example, $1.1d = 660$

OR

considers a specific $C_n = 6000$, for example, $C_2 = 6000$ ***(A1)***

$6000(1.1)^{2-1} - 10d\left((1.1)^{2-1} - 1\right) = 6000$

attempts to solve their $C_n = 6000$ for d ***(M1)***

$6600 - d = 6000$

THEN

$d = 600$ ***A1***

[3 marks]

To model predicted numbers of trout, the lake owner has been using sequences generated by

$u_{n+1} = ru_n - d$, where $d,\ r \in \mathbb{R}^+$ and $r \neq 1$.

(g) Use mathematical induction to prove that $u_n = u_1 r^{n-1} - \frac{d(r^{n-1}-1)}{r-1}$, for $n \in \mathbb{Z}^+$. [7]

Markscheme

$n = 1\ :\ \text{LHS} = u_1$ and $\text{RHS} = \left(u_1 r^{1-1} - \frac{d(r^{1-1}-1)}{r-1}\right) = u_1 r^0 - \frac{d(r^0-1)}{r-1} = u_1$ ***R1***

$\text{LHS} = \text{RHS}$ and so true for $n = 1$

Note: Award ***R0*** for considering $n = 2$.

Subsequent marks after this ***R1*** are independent of this mark and can be awarded.

assume true for $n = k$ $\left(u_k = u_1 r^{k-1} - \frac{d(r^{k-1}-1)}{r-1}\right)$ ***M1***

Note: The assumption of truth must be apparent. Do not award ***M1*** for statements such as "let $n = k$" or "assume that $n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

attempts to substitute for u_k in $u_{k+1} = ru_k - d$ ***M1***

$= r\left(u_1 r^{k-1} - \frac{d(r^{k-1}-1)}{r-1}\right) - d$

$= u_1 r^k - \frac{dr(r^{k-1}-1)}{r-1} - \frac{d(r-1)}{r-1}$ $\left(= u_1 r^k - \frac{dr^k - dr}{r-1} - d\right)$ ***A1***

$= u_1 r^k - \frac{dr^k + dr - dr - d}{r-1}$ $\left(= u_1 r^k - \frac{dr^k - d}{r-1}\right)$ ***A1***

$= u_1 r^k - \frac{d(r^k-1)}{r-1}$ ***A1***

since true for $n = 1$ and true for $n = k + 1$ if true for $n = k$, hence true for all $n (\in \mathbb{Z}^+)$ ***R1***

Note: To obtain the final ***R1***, four of the previous six marks must have been awarded.

[7 marks]

20. [Maximum mark: 28] 24N.3.AHL.TZ0.2

A polynomial is said to be palindromic if the sequence of its coefficients remains the same in reverse. This question asks you to investigate some properties and solutions of palindromic polynomial equations.

In parts (a) and (b), consider quadratic equations of the form $ax^2 + bx + a = 0$, where $a \neq 0$.

The sequence of coefficients, $\{a,\ b,\ a\}$, remains the same in reverse.

The following table shows three palindromic quadratic equations and their sequence of coefficients.

Palindromic quadratic equation	Sequence of coefficients
$2x^2 - 5x + 2 = 0$	$\{2, -5, 2\}$
$x^2 + 4x + 1 = 0$	$\{1, 4, 1\}$
$x^2 + 1 = 0$	$\{1, 0, 1\}$

The quadratic equation $2x^2 - 5x + 2 = 0$ has roots 2 and $\frac{1}{2}$.

These roots form a "reciprocal pair", since one root is the reciprocal of the other.

(a.i) Determine the roots of $x^2 + 4x + 1 = 0$.

Give these roots in the form $s \pm \sqrt{t}$, where $s \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. [3]

Markscheme

attempts to solve *(M1)*

EITHER

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$

OR

$(x+2)^2 - 3 \quad (= 0)$

OR

$\left(x + 2 + \sqrt{3}\right)\left(x + 2 - \sqrt{3}\right) \quad (= 0)$

THEN

$x = -2 \pm \sqrt{3}$ *A1A1*

Note: Award *A1* for $-2 + \sqrt{3}$ and *A1* for $-2 - \sqrt{3}$.

[3 marks]

(a.ii) Hence, or otherwise, show that these roots form a reciprocal pair. [2]

Markscheme

METHOD 1

attempts to find their $\frac{1}{-2-\sqrt{3}} \times \frac{-2+\sqrt{3}}{-2+\sqrt{3}}$ ***(M1)***

Note: Only award ***(M1)*** if the roots are in the form $s \pm \sqrt{t}$.

$= \frac{-2+\sqrt{3}}{(-2)^2-\left(\sqrt{3}\right)^2}$ and $(-2)^2 - \left(\sqrt{3}\right)^2 = 1$ ***A1***

hence the roots form a reciprocal pair ***AG***

Note: Award as above for attempting to show that $\frac{1}{-2+\sqrt{3}} = -2 - \sqrt{3}$.

METHOD 2

considers the product of their roots ***(M1)***

Note: Only award ***(M1)*** if the roots are in the form $s \pm \sqrt{t}$.

$\left(-2 + \sqrt{3}\right)\left(-2 - \sqrt{3}\right) = 1$ ***A1***

hence the roots form a reciprocal pair ***AG***

METHOD 3

considers $\frac{c}{a}$ $\left(= \frac{1}{1}\right)$ ***(M1)***

the product of roots is 1 ***A1***

hence the roots form a reciprocal pair ***AG***

[2 marks]

(b) Show that the complex roots of $x^2 + 1 = 0$ form a reciprocal pair. [2]

Markscheme

roots are $\pm\mathrm{i}$ $\left(= \mathrm{i}, \frac{1}{\mathrm{i}},\ = \mathrm{cis}\left(\pm\frac{\pi}{2}\right), = \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}}\right)$ ***(A1)***

EITHER

$\frac{1}{-\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = \mathrm{i}$ OR $\frac{1}{-\mathrm{i}} = \mathrm{i}$ ***A1***

Note: Award as above for showing that $\frac{1}{\mathrm{i}} = -\mathrm{i}$ OR stating that $\frac{1}{\mathrm{i}} = -\mathrm{i}$.

OR

$\mathrm{i} \times -\mathrm{i} = 1$ OR $\mathrm{i} \times \frac{1}{\mathrm{i}} = 1$ ***A1***

OR

the product of roots is 1 ***A1***

THEN

hence the roots form a reciprocal pair ***AG***

[2 marks]

Let $p(x) = ax^2 + bx + a$, where $a \neq 0$.

(c) Verify that $p(x) = x^2 p\left(\frac{1}{x}\right)$ where $x \neq 0$. [2]

Markscheme

$p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$ ***(A1)***

$x^2 p\left(\frac{1}{x}\right) = x^2\left(\frac{a}{x^2} + \frac{b}{x} + a\right)$ ***A1***

$= ax^2 + bx + a$

$= p(x)$ ***AG***

Note: Award as above for showing that $\frac{p(x)}{x^2} = p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$.

[2 marks]

In parts (d) and (e), you may assume the result that a polynomial, $p(x)$, of degree n is palindromic if and only if $p(x) = x^n p\left(\frac{1}{x}\right)$.

(d) Use $p(x) = x^n p\left(\frac{1}{x}\right)$ to show that if $\alpha \neq 0$ is a root of $p(x) = 0$, then $\frac{1}{\alpha}$ is also a root. [2]

Markscheme

METHOD 1

$p(\alpha) = 0 \Rightarrow \alpha^n p\left(\frac{1}{\alpha}\right) = 0$ ***A1***

EITHER

$\alpha^n \neq 0$ (as $\alpha \neq 0$) $\Rightarrow p\left(\frac{1}{\alpha}\right) = 0$ ***R1***

OR

$\alpha \neq 0$ (hence $\alpha^n \neq 0$) $\Rightarrow p\left(\frac{1}{\alpha}\right) = 0$ ***R1***

THEN

so $\frac{1}{\alpha}$ is also a root ***AG***

Note: The ***R1*** is dependent on the ***A1***.

METHOD 2

$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p\left(\frac{1}{\frac{1}{\alpha}}\right)$

$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p(\alpha)$ ***A1***

for $\alpha \neq 0$ (seen anywhere), $p(\alpha) = 0 \Rightarrow \left(\frac{1}{\alpha}\right)^n p(\alpha) = 0$ and so $p\left(\frac{1}{\alpha}\right) = 0$ ***R1***

so $\frac{1}{\alpha}$ is also a root ***AG***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

Let $f(x) = p(x)q(x)$, where p and q are palindromic polynomials of degree n and m respectively.

(e) Show that f is a palindromic polynomial. [4]

Markscheme

METHOD 1

$p(x) = x^n p\left(\frac{1}{x}\right)$

$q(x) = x^m q\left(\frac{1}{x}\right)$ ***(A1)***

attempts to substitute for $p(x)$ and $q(x)$ into $f(x) = p(x)q(x)$ ***(M1)***

$f(x) = x^n p\left(\frac{1}{x}\right) x^m q\left(\frac{1}{x}\right) \quad \left(= x^{n+m} p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right)\right)$ ***A1***

$= x^{n+m} f\left(\frac{1}{x}\right)$ (and $n + m$ is the degree of f) ***A1***

hence f is a palindromic polynomial. ***AG***

METHOD 2

$p\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}$

$q\left(\frac{1}{x}\right) = \frac{q(x)}{x^m}$ ***(A1)***

attempts to substitute for $p\left(\frac{1}{x}\right)$ and $q\left(\frac{1}{x}\right)$ into $f\left(\frac{1}{x}\right) = p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right)$ ***(M1)***

$f\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}\frac{q(x)}{x^m} \quad \left(= \frac{p(x)q(x)}{x^{n+m}}\right)$ ***A1***

$x^{n+m}f\left(\frac{1}{x}\right) = p(x)q(x)$ and $f(x) = p(x)q(x)$ (and $n+m$ is the degree of f) ***A1***

hence f is a palindromic polynomial. ***AG***

METHOD 3

for example, $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_{n-1}x + a_n$ AND

$q(x) = b_mx^m + b_{m-1}x^{m-1} + \ldots + b_{m-1}x + b_m$ ***A1***

attempts to multiple out their $p(x)q(x)$ **(M1)**

Note: Only award ***(M1)*** for polynomials of degree n and m respectively with different corresponding coefficients.

$p(x)q(x) = a_nb_mx^{m+n} + (a_{n-1}b_m + a_nb_{m-1})x^{m+n-1} + \ldots + (a_{n-1}b_m + a_nb_{m-1})x + a_nb_m$ ***A1A1***

Note: Award ***A1*** for the first and second terms and ***A1*** for the penultimate and last terms.

hence f is a palindromic polynomial ***AG***

[4 marks]

Consider the palindromic polynomial $f(x) = x^4 + 2x^3 - x^2 + 2x + 1$.

This polynomial can be expressed in the form

$f(x) = (x^2 + ux + 1)(x^2 + vx + 1)$, where $u, v \in \mathbb{Z}$ and $u < v$.

(f) By forming and solving an appropriate system of equations in u and v, determine the value of u and the value of v. [6]

Markscheme

METHOD 1

attempts to expand the product of two quadratic factors ***(M1)***

EITHER

$x^4 + ux^3 + vx^3 + uvx^2 + 2x^2 + ux + vx + 1$ ***(A1)***

OR

$x^4 + (u+v)x^3 + (uv+2)x^2 + (u+v)x + 1$ ***(A1)***

THEN

equates coefficients and forms a system of two equations in u and v ***(M1)***

$u+v=2$ and $uv+2=-1 \quad (uv=-3)$ ***A1***

attempts to solve their system of two equations ***(M1)***

$u=-1,\ v=3$ OR $(f(x)=)\ \left(x^2-x+1\right)\left(x^2+3x+1\right)$ ***A1***

Note: Candidates must attempt to form and solve a system of equations in u and v. Hence, award no marks for working backwards from either decimal or exact values of roots given by a GDC.

Award ***A0*** for $u=-1,\ v=3$ and $u=3,\ v=-1$ both specified.

METHOD 2

attempts to divide one of $\left(x^2+ux+1\right)$ or $\left(x^2+vx+1\right)$ into $f(x)$ ***(M1)***

EITHER

for example, the quotient is $x^2+(2-u)x+u^2-2u-2$ ***A1***

sets their quotient equal to x^2+vx+1 ***(M1)***

$x^2+(2-u)x+u^2-2u-2=x^2+vx+1$

$u^2-2u-2=1$ and $2-u=v$ ***A1***

attempts to solve their equations for u and then v (or vice versa) ***(M1)***

OR

for example, the remainder is $u\left(-u^2+2u+3\right)x+\left(-u^2+2u+3\right)$ ***A1***

sets their remainder equal to zero ***(M1)***

attempts to solve for u(or v) ***(M1)***

$u=-1,3$ and $u+v=2$ **A1**

THEN

$u=-1,\ v=3$ OR $(f(x)=)\ \left(x^2-x+1\right)\left(x^2+3x+1\right)$ ***A1***

Note: Award ***A0*** for $u=-1,\ v=3$ and $u=3,\ v=-1$ both specified.

METHOD 3

considers $f(1)$, for example ***(M1)***

$(u+2)(v+2)=5$ ***A1***

considers $f(2)$, for example ***(M1)***

$(2u+5)(2v+5)=33$ ***A1***

attempts to solve their system of two equations ***(M1)***

$u=-1,\ v=3$ OR $(f(x)=)\ \left(x^2-x+1\right)\left(x^2+3x+1\right)$ ***A1***

Note: Award ***A0*** for $u=-1,\ v=3$ and $u=3,\ v=-1$ both specified.

[6 marks]

(g) Hence, find all the exact complex and purely real roots of $x^4+2x^3-x^2+2x+1=0$. [3]

Markscheme

attempts to solve $x^4+2x^3-x^2+2x+1=0$ ***(M1)***

Note: Award ***(M1)*** for attempting to solve one of their quadratic equations.

$\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{3}{2}\pm\frac{\sqrt{5}}{2}$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$ and ***A1*** for $-\frac{3}{2}\pm\frac{\sqrt{5}}{2}$.

[3 marks]

Consider the palindromic polynomial equation

$x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_2x^2+a_1x+1=0$, where n is odd.

(h) Show that -1 is always a root of this equation. [4]

Markscheme

METHOD 1

consider $p(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_2x^2+a_1x+a_0\ (=0)$

attempts to find $p(-1)$ ***(M1)***

$p(-1)=(-1)^n+a_1(-1)^{n-1}+a_2(-1)^{n-2}+\ldots+a_2(-1)^2+a_1(-1)+1$ ***(A1)***

$=(-1)+a_1-a_2+\ldots+a_2-a_1+1$ ***A1***

(since n is odd), $p(x)$ has

EITHER

$n+1$ terms so all terms cancel giving $p(-1)=0$ ***R1***

OR

an even number of terms so all terms cancel giving $p(-1) = 0$ ***R1***

THEN

hence -1 is always a root ***AG***

METHOD 2

considers the product of roots of a polynomial equation of degree n ***(M1)***

$\frac{(-1)^n a_0}{a_n}$

product of roots is $(-1)^n$ (seen anywhere) ***A1***

(for n odd) there are $\frac{n-1}{2}$ reciprocal pairs of roots ***A1***

product of each reciprocal pair is 1 and $(-1)^n = -1$ (since n is odd) ***R1***

hence -1 is always a root ***AG***

METHOD 3

consider $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 \ (= 0)$

considers $p(x) = x^n p\left(\frac{1}{x}\right)$ with $x = -1$ ***(M1)***

$p(-1) = (-1)^n p(-1)$ ***A1***

$p(-1) = -p(-1) \quad (2p(-1) = 0)$ (since n is odd) ***A1***

hence $p(-1) = 0$ ***R1***

hence -1 is always a root ***AG***

[4 marks]

21. [Maximum mark: 27] 24M.3.AHL.TZ1.1

This question considers two possible models for the occurrence of random events in a computer game.

In a new computer game, each time a player performs an action, there is a random chance that the action will be *boosted*, meaning that it provides a benefit to the player.

The designer of this computer game is considering two possible models for when to boost an action.

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1.

(a.i) Find the probability that the first boost occurs on the third action. [2]

Markscheme

$0.9 \times 0.9 \times 0.1$ ***(M1)***

Note: Award ***M1*** for 0.9 seen as part of a multiplication.

$= 0.081$ ***A1***

[2 marks]

(a.ii) Find the probability that at least one boost occurs in the first six actions. [3]

Markscheme

let $A =$number of boosts in the first 6 actions

METHOD 1

Recognition of binomial distribution ***(M1)***

with parameters $n = 6$ and $p = 0.1$ ***(A1)***

$(P(A \geq 1) =)0.469 \ \ (0.468558\ldots)$ ***A1***

Note: Award ***(M1)(A1)A0*** for an answer of 0.114 (from finding $P(A > 1)$).

METHOD 2

$(\mathrm{P}(A \geq 1) =)\ 1 - \mathrm{P}(\text{no boost in first 6})$ ***(M1)***

$= 1 - 0.9^6$ ***(A1)***

$= 0.469\ (0.468558\ldots)$ ***A1***

METHOD 3

$0.1 + 0.9 \times 0.1 + 0.9^2 \times 0.1 + 0.9^3 \times 0.1 + 0.9^4 \times 0.1 + 0.9^5 \times 0.1$ ***(M1)(A1)***

Note: Award ***(M1)*** for at least 3 correct terms in a sum.

$= 0.469\ (0.468558\ldots)$ ***A1***

[3 marks]

(b) Suppose the probability that an action will be boosted is p, where $0 < p < 1$.

(b.i) Explain why the probability that the first boost occurs on the x^{th} action is $p(1-p)^{x-1}$. [1]

Markscheme

EITHER

there will be $x-1$ actions not boosted, followed by one boosted action ***A1***

OR

The x^{th} action is boosted and all previous actions are not boosted ***A1***

OR

$\underbrace{(1-p)\times(1-p)\times\ldots\times(1-p)}_{(x-1)\text{ times}}\times p$ ***A1***

Note: For the **OR** method, the order of the operations must be clear (accept also a tree diagram etc.).

THEN

$p(1-p)^{x-1}$ ***AG***

[1 mark]

Let X be the number of actions until the first boost occurs.

(b.ii) Hence, write down an expression, using sigma notation, for $\mathrm{E}(X)$ in terms of x and p. [1]

Markscheme

$\mathrm{E}(X)=\sum_{x=1}^{\infty} xp(1-p)^{x-1}$ ***A1***

[1 mark]

Consider the sum of an infinite geometric sequence, with first term a and common ratio $r(|r|<1)$,

$a+ar+ar^2+ar^3+\ldots=\frac{a}{1-r}$.

(c.i) By differentiating both sides of the above equation with respect to r, find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r. [4]

Markscheme

LHS:

$a+2ar+3ar^2+\ldots$ ***A1***

Note: Award ***A1*** for writing LHS $=\Sigma ar^n$ and differentiating to get $=\Sigma anr^{n-1}$.

RHS:

attempt to differentiate $a(1-r)^{-1}$ or use quotient rule ***(M1)***

$\frac{a}{(1-r)^2}$ ***A1***

$\sum_{n=1}^{\infty} nar^{n-1} = \frac{a}{(1-r)^2}$ ***A1***

Note: To award the final ***A1*** "$\sum_{n=1}^{\infty} nar^{n-1}$" must be seen either as part of the LHS working or as part of the final answer and all previous marks must be awarded.

[4 marks]

(c.ii) Hence, show that $\mathrm{E}(X) = \frac{1}{p}$. [2]

Markscheme

recognition that $a = p$ and $r = 1 - p$ (and $x = n$) ***(M1)***

$\mathrm{E}(X) = \frac{p}{(1-(1-p))^2}$ ***A1***

Note: Award ***A0 FT*** if their answer does not lead to the ***AG***.

$= \frac{1}{p}$ ***AG***

[2 marks]

It can be shown that $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

(d) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ when $p = 0.1$. [2]

Markscheme

$\mathrm{E}(X) = 10,\ \ \mathrm{Var}(X) = 90$ ***A1A1***

[2 marks]

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

(e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Markscheme

$0.8 \times 0.6 \times 0.6$ ***(M1)A1***

Note: Award ***M1*** for evidence of P("not boosted") changing, e.g. a labelled tree diagram and/or 0.8 AND 0.6 seen.

$= 0.288$ ***AG***

[2 marks]

Let Y be the number of actions until the first boost occurs.

(f) Explain why $Y \leq 5$. [1]

Markscheme

after four actions that are not boosted, the probability that the next action is boosted is 1 (and $p = 1$ is a certainty) ***R1***

Note: Accept "when $Y = 5$, the probability of a boost is 1".

Do not accept "on the 5^{th} action the probability of a boost is 1", unless there is reference to the previous actions being not boosted.

[1 mark]

The following table shows the probability distribution of Y.

y	1	2	3	4	5
$\text{P}(Y = y)$	0.2	m	0.288	n	0.0384

(g.i) Find the value of m and the value of n. [2]

Markscheme

$m = 0.32,\ n = 0.1536\ \ (0.154)$ ***A1A1***

[2 marks]

(g.ii) Show that $\text{E}(Y) = 2.5104$. [2]

Markscheme

attempt to multiply outcomes by probabilities and find the sum ***(M1)***

$0.2 + 2 \times 0.32 + 3 \times 0.288 + 4 \times 0.1536 + 5 \times 0.0384$ ***A1***

Note: The ***A1*** can only be awarded if m and n are correct and the exact value of n is used.

$= 2.5104$ ***AG***

[2 marks]

(g.iii) Find $\mathrm{Var}(Y)$. [2]

Markscheme

$0.2 + 2^2 m + 3^2 \times 0.288 + 4^2 \times n + 5^2 \times 0.0384 - 2.5104^2$ ***(A1)***

$= 1.19\ (1.18749\ldots)$ ***A1***

Note: Award ***A1A1*** for a correct answer found using the GDC.

[2 marks]

(h.i) Use the expression given in (c)(ii) to find the value of p for which $\mathrm{E}(X) = \mathrm{E}(Y)$. [1]

Markscheme

$\left(p = \frac{1}{2.5104}\right) = 0.398\ (0.398342\ldots)$ ***A1***

[1 mark]

(h.ii) Find $\mathrm{Var}(X)$ for this value of p. [1]

Markscheme

$\mathrm{Var}(X) = 3.79\ (3.79170\ldots)$ ***A1***

Note: Award ***A1*** for an answer of $3.80\ (3.80040\ldots)$ (from using their 3sf answer from part (i).)

Condone 3.8.

[1 mark]

(h.iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]

Markscheme

since $\mathrm{Var}(X) > \mathrm{Var}(Y)$, 2^{nd} model provides more consistent experience ***R1***

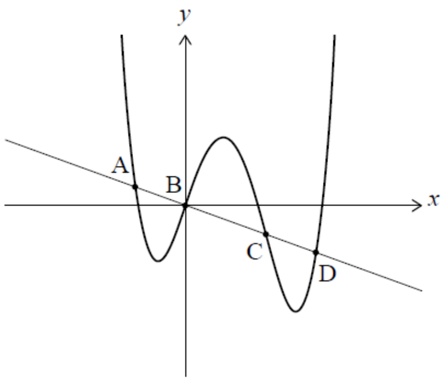
Note: Only award ***R1 FT*** if both their variances are positive.

[1 mark]

22. [Maximum mark: 28] 24M.3.AHL.TZ1.2

This question investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.

The curve $y = x^4 - 3x^3 + 3x$ has points of inflexion at B and C. The line passing through B and C intersects the curve again at points A and D. This is shown in the following graph.



(a) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [3]

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 + 3$ ***(M1)(A1)***

Note: Award ***M1*** for at least two correct terms.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 18x$ ***A1***

[3 marks]

(b) Find the coordinates of B and C. [4]

Markscheme

valid attempt to find x-coordinates (e.g. solving $12x^2 - 18x = 0$ or graphing $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 + 3$) ***(M1)***

$x = 0,\ 1.5\ \left(\frac{3}{2}\right)$ ***(A1)***

point B is $(0,\ 0)$ ***A1***

point C is $(1.5,\ -0.5625)$ $\left((1.5,\ -0.563),\ \left(\frac{3}{2},\ -\frac{9}{16}\right)\right)$ ***A1***

Note: Award ***M0A0A1A0*** for an unsupported answer of point B is $(0,\ 0)$.

[4 marks]

(c) Show that the equation of the line through B and C is $y = -0.375x$. [2]

Markscheme

y-intercept $= 0$ (as line passes through $(0,\ 0)$) ***R1***

Note: Award ***R1*** for correctly substituting point B or point C to show that $y = 0$.

Award ***R1*** for $y = -0.375 + 0$.

gradient$= \frac{-0.5625}{1.5}$ $\left(= \frac{-\frac{9}{16}}{\frac{3}{2}}\right)$ ***A1***

Note: Award ***A0FT*** if their answer to (b) doesn't lead to the given answer, but condone $\frac{-0.563}{1.5}$.

$= -0.375$

so equation is $y = -0.375x$ ***AG***

Note: Award at most ***A1R0*** for working backwards to verify points B and C lie on the given line.

[2 marks]

(d) Find, correct to three decimal places, the x-coordinate of D. [2]

Markscheme

$x^4 - 3x^3 + 3x = -0.375x$ ***(M1)***

2.427 ***A1***

Note: Correct answer must be given to three decimal places for the ***A1*** to be awarded.

Award ***(M1)A1*** for $(2.427, \ -0.910)$.

Award ***(M1)A0*** for an unsupported answer of 2.43.

[2 marks]

Now consider the general curve $y = x^4 - mx^3 + nx$, where $m, \ n \in \mathbb{R}$ and $m > 0$.

(e) Find the x-coordinates of the two points of inflexion in terms of m. [3]

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 3mx^2 + n$ ***(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 6mx$ ***(A1)***

$6x(2x - m) = 0$

$x = 0, \ \frac{m}{2}$ ***A1***

Note: Accept $0m$ in place of 0.

[3 marks]

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x-coordinate of point A, and similarly for x_B, x_C and x_D. It is given that $x_A < x_B < x_C < x_D$.

(f.i) Write down the coordinates of B. [1]

Markscheme

point B is $(0,\ 0)$ ***A1***

[1 mark]

(f.ii) Find, in terms of m and n, the coordinates of C. [2]

Markscheme

$y_c = \left(\frac{m}{2}\right)^4 - m\left(\frac{m}{2}\right)^3 + n\left(\frac{m}{2}\right) \quad \left(= \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}\right)$ ***(A1)***

$= -\frac{m^4}{16} + \frac{nm}{2}$ OR $\frac{8nm - m^4}{16}$ OR $\frac{m}{2}\left(n - \frac{m^3}{8}\right)$ OR equivalent simplification ***A1***

Point C is $\left(\frac{m}{2},\ -\frac{m^4}{16} + \frac{nm}{2}\right)$

Note: Award the second ***A1*** if the simplified y-coordinate is seen here or in part (g).

[2 marks]

(g) Show that the equation of the line through B and C is $y = \left(-\frac{m^3}{8} + n\right)x$. [2]

Markscheme

attempt to divide their y_c by their x_c OR substitute into $\frac{y_2 - y_1}{x_2 - x_1}$ ***(M1)***

e.g. $\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \div \frac{m}{2}$

$\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \times \frac{2}{m}$ **OR** equivalent manipulation leading to given answer ***A1***

Note: Award ***A0 FT*** if their answer does not lead to the ***AG***.

$= -\frac{m^3}{8} + n$ and y-intercept $= 0$,

so equation is $y = \left(-\frac{m^3}{8} + n\right)x$ ***AG***

Note: Award at most ***M1A0*** for working backwards to verify points B and C lie on the given line.

[2 marks]

(h) Show that $x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$. [7]

Markscheme

METHOD 1

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \frac{m^3}{8}x = 0$

recognizing their $(x - x_C)$ is a factor of this expression ***(M1)***

$x\left(x - \frac{m}{2}\right)\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)\ (= 0)$ OR equivalent ***A1M1A1***

Note: Award ***A1*** for $x\left(x - \frac{m}{2}\right)$, ***M1A1*** for $\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)$.

If a candidate divides by x without justification, do not award the first ***A1***, but all subsequent marks can still be awarded.

use of quadratic formula to find roots, x_A (and x_D), of the quadratic ***M1***

$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2}$ OR equivalent ***A1***

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$ ***AG***

Note: Award ***(M1)(M0)A0M0A0M0A0*** for attempting to verify the given answer satisfies

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$.

METHOD 2

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \frac{m^3}{8}x = 0$

$x\left(x^3 - mx^2 + \frac{m^3}{8}\right) = 0$

Recognise that their x_C is a root of this equation ***(M1)***

Attempt to find sum and product of roots ***(M1)***

$x_A + x_D + \frac{m}{2} = m$ AND $x_A \times x_D \times \frac{m}{2} = -\frac{m^3}{8}$ ***A1***

$x_A \times \left(\frac{m}{2} - x_A\right) \times \frac{m}{2} = -\frac{m^3}{8}$

${x_A}^2 - \frac{m}{2}x_A - \frac{m^2}{4} = 0$ ***A1***

use of quadratic formula to find roots, x_A (and x_D), of the quadratic ***M1***

$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2}$ OR equivalent ***A1***

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$ ***AG***

[7 marks]

(i) Hence, find the exact value of $\frac{x_B - x_A}{x_C - x_B}$. [2]

Markscheme

$\frac{0 - \left(\frac{m}{4} - \frac{m}{4}\sqrt{5}\right)}{\frac{m}{2} - 0}$ ***(A1)***

$= \frac{\sqrt{5} - 1}{2}$ ***A1***

Note: Answer must be exact.

[2 marks]

23. [Maximum mark: 24] 24M.3.AHL.TZ2.1

If two functions $f(x)$ and $g(x)$ are differentiable, then their product is differentiable and the two functions satisfy the product rule: $(f(x)g(x))\prime = f(x)g\prime(x) + g(x)f\prime(x)$.

In this question, you will meet examples of pairs of differentiable functions, $f(x)$ and $g(x)$, that also satisfy $(f(x)g(x))\prime = f\prime(x)g\prime(x)$.

In part (a), consider $f(x) = \frac{1}{(2-x)^2}$, where $x \in \mathbb{R}, x \neq 2$, and $g(x) = x^2$, where $x \in \mathbb{R}$.

(a.i) Find an expression for $f\prime(x)$. [2]

Markscheme

attempts chain rule differentiation to find $f\prime(x)$ ***(M1)***

$f\prime(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3}\right)$ ***A1***

Note: Award ***(M1)*** for attempting chain rule differentiation on $\left(4-4x+x^2\right)^{-1}$ or attempting quotient rule differentiation on $\frac{1}{(4-4x+x^2)}\left(=\frac{1}{(2-x)^2}\right)$.

Award ***A1*** for $f\prime(x)=\frac{2}{(2-x)^3}\left(=(-1)(-2)(2-x)^{-3}\right)$.

[2 marks]

(a.ii) Show that $f\prime(x)g\prime(x)=\frac{4x}{(2-x)^3}$. [2]

Markscheme

$g\prime(x)=2x$ ***(A1)***

$f\prime(x)g\prime(x)=\left(2(2-x)^{-3}\right)(2x)\left(=\frac{2(2x)}{(2-x)^3}\right)$ (or equivalent) ***A1***

$=\frac{4x}{(2-x)^3}$ ***AG***

[2 marks]

(a.iii) Show that $f(x)g\prime(x)+g(x)f\prime(x)=\frac{4x}{(2-x)^3}$. [4]

Markscheme

Note: Award a maximum of ***(M1)A1(M1)A0FT*** from parts (a) (i) and (ii).

substitutes $f(x),\ g(x)$ and their $g\prime(x),\ f\prime(x)$ into the given expression ***(M1)***

EITHER

$f(x)g\prime(x)+g(x)f\prime(x)=2x(2-x)^{-2}+2x^2(2-x)^{-3}$ ***A1***

Note: Award ***A1*** if $f(x)g\prime(x)=2x(2-x)^{-2}$ and $g(x)f\prime(x)=2x^2(2-x)^{-3}$ are stated separately.

attempts to factorise their expression ***(M1)***

$=2x(2-x)^{-3}((2-x)+x)$ ***A1***

OR

$f(x)g\prime(x)+g(x)f\prime(x)=\frac{2x}{(2-x)^2}+\frac{2x^2}{(2-x)^3}$ ***A1***

Note: Award ***A1*** if $f(x)g\prime(x)=\frac{2x}{(2-x)^2}$ and $g(x)f\prime(x)=\frac{2x^2}{(2-x)^3}$ are stated separately.

attempts to form an expression with a common denominator ***(M1)***

Note: Award *(M1)* for $(2-x)^2(2-x)^3$ as a common denominator.

$= \frac{2x(2-x)}{(2-x)^3} + \frac{2x^2}{(2-x)^3} \left(= \frac{4x-2x^2+2x^2}{(2-x)^3}\right)$ ***A1***

THEN

$= \frac{4x}{(2-x)^3}$ ***AG***

Note: Award marks as appropriate for attempting to find the derivative of $f(x)g(x) = \frac{x^2}{(2-x)^2}$ (or equivalent).

[4 marks]

In parts (b) and (c), consider two non-constant functions, $f(x)$ and $g(x)$, where $f(x) > 0$ and $g(x) \neq g\prime(x)$.

(b) By rearranging the equation $f(x)g\prime(x) + g(x)f\prime(x) = f\prime(x)g\prime(x)$, show that $\frac{f\prime(x)}{f(x)} = \frac{g\prime(x)}{g\prime(x)-g(x)}$. [2]

Markscheme

METHOD 1

$f\prime(x)g\prime(x) - g(x)f\prime(x) = f(x)g\prime(x)$ ***(A1)***

$(f\prime(x)g\prime(x) - g(x)f\prime(x) - f(x)g\prime(x) = 0)$

$f\prime(x)(g\prime(x) - g(x)) = f(x)g\prime(x)$ $(f\prime(x)(g(x) - g\prime(x)) = -f(x)g\prime(x))$ ***A1***

$\frac{f\prime(x)}{f(x)} = \frac{g\prime(x)}{g\prime(x)-g(x)}$ ***AG***

Note: Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 2

$g\prime(x) = \frac{f\prime(x)g\prime(x)}{f(x)} - \frac{g(x)f\prime(x)}{f(x)}$ ***(A1)***

$g\prime(x) = \frac{f\prime(x)}{f(x)}(g\prime(x) - g(x))$ ***A1***

$\frac{f\prime(x)}{f(x)} = \frac{g\prime(x)}{g\prime(x)-g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.

Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 3

$g\prime(x) = \frac{f(x)g\prime(x)}{f\prime(x)} + g(x)$ ***(A1)***

$\frac{f(x)}{f\prime(x)} = \frac{g\prime(x)-g(x)}{g\prime(x)}$ ***A1***

$\frac{f\prime(x)}{f(x)} = \frac{g\prime(x)}{g\prime(x)-g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.

Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 4

$\frac{f(x)}{f\prime(x)} + \frac{g(x)}{g\prime(x)} = 1$ ***(A1)***

$\frac{f(x)}{f\prime(x)} = \frac{g\prime(x)-g(x)}{g\prime(x)}$ ***A1***

$\frac{f\prime(x)}{f(x)} = \frac{g\prime(x)}{g\prime(x)-g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.

Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

[2 marks]

(c) Hence, by integrating both sides of $\frac{f\prime(x)}{f(x)} = \frac{g\prime(x)}{g\prime(x)-g(x)}$, show that $f(x) = A\mathrm{e}^{\left(\int \frac{g\prime(x)}{g\prime(x)-g(x)}\mathrm{d}x\right)}$, where A is an arbitrary positive constant. [2]

Markscheme

METHOD 1

Note: Condone the absence of '$\mathrm{d}\ x$' and the modulus sign throughout.

Only award the second ***A*** mark if the constant of integration has been dealt with correctly.

EITHER

$\ln f(x) = \int \frac{g\prime(x)}{g\prime(x)-g(x)}\ \mathrm{d}\ x\ (+C)$ ***A1***

Note: Condone the absence of '$+C$' when awarding the first ***A*** mark.

$f(x) = \mathrm{e}^{\left(\int \frac{g\prime(x)}{g\prime(x)-g(x)}\mathrm{d}x\right)}\mathrm{e}^{C}\ \left(f(x) = \mathrm{e}^{\left(\int \frac{g\prime(x)}{g\prime(x)-g(x)}\mathrm{d}x+C\right)}\right)$ ***A1***

Note: Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g\prime(x)}{g\prime(x)-g(x)}\mathrm{d}x\right)}$, where $A = \mathrm{e}^{C}$.

OR

$\ln f(x)\ (+C) = \int \frac{g'(x)}{g'(x)-g(x)}\ \mathrm{d}\ x$ ***A1***

Note: Condone the absence of '$+C$' when awarding the first ***A*** mark.

$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} \mathrm{e}^{-C} \left(f(x)\mathrm{e}^{C} = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)},\ f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x - C\right)} \right)$ ***A1***

Note: Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$, where $A = \mathrm{e}^{-C}$.

THEN

$f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$ ***AG***

METHOD 2

Note: Condone the absence of '$\mathrm{d}\ x$' throughout.

$f'(x) - \frac{g'(x)}{g'(x)-g(x)} f(x) = 0$

Integrating factor: $\mathrm{e}^{\left(-\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$ ***A1***

Note: Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} = C$.

$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\mathrm{e}^{\left(-\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} \right] = 0 \Rightarrow f(x)\mathrm{e}^{\left(-\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} = A$ ***A1***

Note: Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} = C$

$f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$ ***AG***

[2 marks]

The result from part (c) can be used to find pairs of functions, $f(x)$ and $g(x)$, which satisfy **both** of the following:

$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$ and $(f(x)g(x))' = f'(x)g'(x)$.

In parts (d) and (e), use the result in part (c) with $A = 1$.

(d) Consider $g(x) = xe^x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations. [5]

Markscheme

$g'(x) = xe^x + e^x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\frac{g'(x)}{g'(x)-g(x)}$ ***(M1)***

$= \frac{xe^x+e^x}{e^x} \left(= \frac{e^x(x+1)}{e^x}\right)$

$= x + 1$ (as $e^x \neq 0$) ***(A1)***

attempts to integrate their $\frac{g'(x)}{g'(x)-g(x)}$ ***(M1)***

$\int (x+1)\ \mathrm{d}\ x = \frac{1}{2}x^2 + x(+C)$

$f(x) = e^{\left(\frac{1}{2}x^2+x\right)}$ ***A1***

Note: Award ***A0*** for $f(x) = e^{\left(\frac{1}{2}x^2+x+C\right)}$ (or equivalent expressed with an arbitrary constant).

[5 marks]

(e) Consider $g(x) = \sin x + \cos x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations over the domain $0 < x < \pi$.

Give your answer in the form $f(x) = \sqrt{e^x h(x)}$, where $h(x)$ is a function to be determined. [7]

Markscheme

$g'(x) = \cos x - \sin x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\frac{g'(x)}{g'(x)-g(x)}$ ***(M1)***

$= \frac{\cos x - \sin x}{\cos x - \sin x - \sin x - \cos x} \left(= \frac{\sin x - \cos x}{2\sin x}\right)$

$= \frac{1}{2} - \frac{1}{2}\cot x$ (as $\sin x \neq 0$) OR $= \frac{1}{2} - \frac{1}{2}\frac{\cos x}{\sin x}$ (as $\sin x \neq 0$) ***A1***

$f(x) = e^{\int\left(\frac{1}{2}-\frac{1}{2}\cot x\right)\mathrm{d}x}$

attempts to find the indefinite integral of $(\pm k)\cot x$ OR $(\pm k)\frac{\cos x}{\sin x}$ ***(M1)***

Note: As $|\sin x| = \sin x$ for $0 < x < \pi$, condone the presence or omission of the modulus sign throughout a candidate's solution.

Condone the presence of an arbitrary constant except when awarding the final ***A*** mark.

$\int\left(\frac{1}{2} - \frac{1}{2}\cot x\right)\mathrm{d}\ x = \frac{x}{2} - \frac{1}{2}\ln|\sin x|(+C)\ \left(= \frac{1}{2}(x - \ln|\sin x|(+C))\right)$ ***A1***

$f(x) = e^{\frac{x}{2}} e^{-\frac{1}{2} \ln |\sin x|} \left(e^C\right)$

$= e^{\frac{x}{2}} e^{\ln \sqrt{\frac{1}{\sin x}}} \left(e^C\right) \left(= e^{\frac{x}{2}} e^{\frac{1}{2} \ln \left(\frac{1}{\sin x}\right)} \left(e^C\right), \; = \sqrt{e^{x - \ln (\sin x)}} \left(e^C\right)\right)$ ***A1***

$= e^{\frac{x}{2}} \sqrt{\frac{1}{\sin x}}$

$= \sqrt{e^x \text{ cosec } x} \left(= \sqrt{\frac{e^x}{\sin x}}\right)$ (where $h(x) = \frac{1}{\sin x}$) ***A1***

[7 marks]

24. [Maximum mark: 31] 24M.3.AHL.TZ2.2

This question asks you to find the probability of graphs of randomly generated quadratic functions having a specified number of x-intercepts.

In parts (a) – (f), consider quadratic functions, $f(x) = ax^2 + bx + c$, whose coefficients, a, b and c, are randomly generated in turn by rolling an unbiased six-sided die three times and reading off the value shown on the uppermost face of the die.

For example, rolling a $2, \; 3$ and 5 in turn generates the quadratic function $f(x) = 2x^2 + 3x + 5$.

(a) Explain why there are 216 possible quadratic functions that can be generated using this method. [1]

Markscheme

6^3 OR $6 \times 6 \times 6$ ***A1***

Note: Accept a labelled diagram that clearly illustrates correct application of the multiplication principle leading to 216.

$= 216$ ***AG***

[1 mark]

(b) The set of coefficients, $a = 1, \; b = 4$ and $c = 4$, is randomly generated to form the quadratic function $f(x) = x^2 + 4x + 4$.

Verify that this graph of f has only one x-intercept. [2]

Markscheme

EITHER

attempts to find Δ ***(M1)***

$\Delta\left(= \left(4^2 - 4(1)(4)\right)\right) = 0$ ***A1***

OR

attempts to solve $x^2 + 4x + 4 = 0$ ***(M1)***

$\left((x+2)^2 = 0 \Rightarrow\right) x = -2$ ***A1***

OR

attempts to express $x^2 + 4x + 4(= 0)$ as a perfect square ***(M1)***

$(x + 2)^2(= 0)$ is a perfect square ***A1***

OR

a graph of $y = x^2 + 4x + 4$ with the vertex touching the x-axis at $x = -2$ ***A2***

THEN

graph of f has only one x-intercept ***AG***

[2 marks]

(c) By considering the discriminant, or otherwise, show that the probability of the graph of such a randomly generated quadratic function having only one x-intercept is $\frac{5}{216}$. [6]

Markscheme

Note: In parts (c) – (f), $(a,\ b,\ c) = (1,\ 2,\ 1)$, for example, represents an ordered 3-tuple $a = 1,\ b = 2$ and $c = 1$.

recognizes that $b^2 - 4ac = 0$ (or equivalent) ***(M1)***

EITHER

attempts to use $\frac{b^2}{ac} = 4\left(\frac{b^2}{4} = ac\right)$ ***(M1)***

determines one value of b from $b = 2, 4$ or 6 only (seen anywhere) OR one value of ac from $ac = 1,\ 4$ or 9 only (seen anywhere) ***(A1)***

OR

attempts to find a possible value of b ***(M1)***

determines one value of b from $b = 2,\ 4$ or 6 only (seen anywhere) ***(A1)***

OR

recognizes that b^2 must be a multiple of 4 OR b must be a multiple of 2 ***(M1)***

determines one value of b from $b = 2,\ 4$ or 6 only (seen anywhere) ***(A1)***

OR

attempts to find a possible value of ac ***(M1)***

determines one value of ac from $ac = 1,\ 4$ or 9 only (seen anywhere) ***(A1)***

THEN

$b = 2$ and $ac = 1$:

$(a,\ b,\ c) = (1,\ 2,\ 1)$ OR 1 possible way OR $\frac{1}{216}$ ***A1***

$b = 4$ and $ac = 4$:

$(a,\ b,\ c) = (1,\ 4,\ 4),\ (4,\ 4,\ 1),\ (2,\ 4,\ 2)$ OR 3 possible ways OR $\frac{3}{216}$ ***A1***

$b = 6$ and $ac = 9$:

$(a,\ b,\ c) = (3,\ 6,\ 3)$ OR 1 possible way OR $\frac{1}{216}$ ***A1***

therefore the required probability is $\frac{1}{216} + \frac{3}{216} + \frac{1}{216}$

$= \frac{5}{216}$ ***AG***

[6 marks]

Now consider randomly generated quadratic functions whose corresponding graphs have two **distinct** x-intercepts.

(d) By considering the discriminant, determine the set of possible values of ac. [3]

Markscheme

Note: In parts (c) – (f), $(a,\ b,\ c) = (1,\ 2,\ 1)$, for example, represents an ordered 3-tuple $a = 1,\ b = 2$ and $c = 1$.

recognizes that $b^2 - 4ac > 0$ (or equivalent eg. $\frac{b^2}{4} > ac$) ***(M1)***

maximum value of b^2 is 36 OR maximum value of ac is 8 ***(A1)***

Note: The above ***(A1)*** is independent of the ***(M1)***.

$ac = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8$ ***A1***

[3 marks]

(e.i) For the case where $ac = 1$, show that there are four quadratic functions whose corresponding graphs have two distinct x-intercepts. [1]

Markscheme

Note: In parts (c) – (f), $(a,\ b,\ c) = (1,\ 2,\ 1)$, for example, represents an ordered 3-tuple $a = 1,\ b = 2$ and $c = 1$.

$ac = 1\ \left(b^2 > 4\right)$

$b = 3,\ 4,\ 5,\ 6$ OR 1×4 (quadratics) OR $6 - 2$ (quadratics) ***A1***

there are four quadratic functions ***AG***

[1 mark]

(e.ii) For the case where $ac = 2$, show that there are eight quadratic functions whose corresponding graphs have two distinct x-intercepts. [2]

Markscheme

Note: In parts (c) – (f), $(a,\ b,\ c) = (1,\ 2,\ 1)$, for example, represents an ordered 3-tuple $a = 1,\ b = 2$ and $c = 1$.

$ac = 2\ \left(b^2 > 8\right)$

$b = 3,\ 4,\ 5,\ 6$ ***(A1)***

Note: Award ***(A1)*** for referencing their result shown in part (e) (i).

EITHER

$(a,\ b,\ c) = (1,\ 3,\ 2),\ (1,\ 4,\ 2),\ (1,\ 5,\ 2),\ (1,\ 6,\ 2),\ (2,\ 3,\ 1),\ (2,\ 4,\ 1),\ (2,\ 5,\ 1),\ (2,\ 6,\ 1)$ ***A1***

Note: Award **A1** for listing the eight quadratic expressions.

OR

2×4 (quadratics) ***A1***

THEN

there are eight quadratics functions ***AG***

[2 marks]

Let p be the probability of the graph of such a randomly generated quadratic function having two distinct x-intercepts.

(f) Using the approach started in part (e), or otherwise, find the value of p. [6]

Markscheme

Note: In parts (c) – (f), $(a,\ b,\ c) = (1,\ 2,\ 1)$, for example, represents an ordered 3-tuple $a = 1,\ b = 2$ and $c = 1$.

METHOD 1

varies ac $(ac \neq 1,\ 2)$ and determines possible values of b such that $\Delta > 0$ ***(M1)***

correctly determines one of the following five cases ***(A1)***

correctly determines a further two of the following five cases ***(A1)***

correctly determines the remaining two cases ***(A1)***

case 1: $ac = 3\ \left(b^2 > 12 \Rightarrow b = 4,\ 5,\ 6\right)$

$(a,\ b,\ c) = (1,\ 4,\ 3),\ (1,\ 5,\ 3),\ (1,\ 6,\ 3),\ (3,\ 4,\ 1),\ (3,\ 5,\ 1),\ (3,\ 6,\ 1)$ OR

6 possible ways OR $\frac{6}{216}$

case 2: $ac = 4\ \left(b^2 > 16 \Rightarrow b = 5,\ 6\right)$

$(a,\ b,\ c) = (1,\ 5,\ 4),\ (1,\ 6,\ 4),\ (2,\ 5,\ 2),\ (2,\ 6,\ 2),\ (4,\ 5,\ 1),\ (4,\ 6,\ 1)$ OR

6 possible ways OR $\frac{6}{216}$

case 3: $ac = 5 \left(b^2 > 20 \Rightarrow b = 5,\ 6\right)$

$(a,\ b,\ c) = (1,\ 5,\ 5),\ (1,\ 6,\ 5),\ (5,\ 5,\ 1),\ (5,\ 6,\ 1)$ OR 4 possible ways OR $\frac{4}{216}$

case 4: $ac = 6 \left(b^2 > 24 \Rightarrow b = 5,\ 6\right)$

$(a,\ b,\ c) = (1,\ 5,\ 6),\ (2,\ 5,\ 3),\ (3,\ 5,\ 2),\ (6,\ 5,\ 1),\ (1,\ 6,\ 6),\ (2,\ 6,\ 3),\ (3,\ 6,\ 2),\ (6,\ 6,\ 1)$

OR 8 possible ways OR $\frac{8}{216}$

case 5: $ac = 8 \left(b^2 > 32 \Rightarrow b = 6\right)$

$(a,\ b,\ c) = (2,\ 6,\ 4),\ (4,\ 6,\ 2)$ OR 2 possible ways OR $\frac{2}{216}$

adds their probabilities ***(M1)***

Note: Award ***(M1)*** for adding at least 3 of their probabilities (denominator 216).

$(p =)\ \frac{4}{216} + \frac{8}{216} + \frac{6}{216} + \frac{6}{216} + \frac{4}{216} + \frac{8}{216} + \frac{2}{216}$

$(= 0.0185\ldots + 0.0370\ldots + 0.0277\ldots + 0.0277\ldots + 0.0185\ldots + 0.0370\ldots + 0.0092\ldots)$

$= \frac{38}{216} \left(= \frac{19}{108},\ = 0.176\right)$ ***A1***

METHOD 2

varies $b^2 (\neq 1,\ 4)$ OR $b (\neq 1,\ 2)$ and determines possible values of ac such that $\Delta > 0$ ***(M1)***

correctly determines one of the following four cases ***(A1)***

correctly determines another case from the following four cases ***(A1)***

correctly determines the remaining two cases ***(A1)***

case 1: $b^2 = 9\ (b = 3)\ (ac = 1,\ 2)$

$(a,\ b,\ c) = (1,\ 3,\ 1),\ (1,\ 3,\ 2),\ (2,\ 3,\ 1)$ OR 3 possible ways OR $\frac{3}{216}$

case 2: $b^2 = 16\ (b = 4)\ (ac = 1,\ 2,\ 3)$

$(a,\ b,\ c) = (1,\ 4,\ 1),\ (1,\ 4,\ 2),\ (2,\ 4,\ 1),\ (1,\ 4,\ 3),\ (3,\ 4,\ 1)$ OR 5 possible ways OR $\frac{5}{216}$

case 3: $b^2 = 25\ (b = 5)\ (ac = 1,\ 2,\ 3,\ 4,\ 5,\ 6)$

$(a,\ b,\ c) = (1,\ 5,\ 1),\ (1,\ 5,\ 2),\ (2,\ 5,\ 1),\ (1,\ 5,\ 3),\ (3,\ 5,\ 1),\ (1,\ 5,\ 4),\ (2,\ 5,\ 2)$

$(4,\ 5,\ 1),\ (1,\ 5,\ 5),\ (5,\ 5,\ 1),\ (1,\ 5,\ 6),\ (2,\ 5,\ 3),\ (3,\ 5,\ 2),\ (6,\ 5,\ 1)$

OR 14 possible ways OR $\frac{14}{216}$

case 4: $b^2 = 36\ (b = 6)\ (ac = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8)$

$(a,\ b,\ c) = (1,\ 6,\ 1),\ (1,\ 6,\ 2),\ (2,\ 6,\ 1),\ (1,\ 6,\ 3),\ (3,\ 6,\ 1),\ (1,\ 6,\ 4),\ (2,\ 6,\ 2),\ (4,\ 6,\ 1)$

$(1,\ 6,\ 5),\ (5,\ 6,\ 1),\ (1,\ 6,\ 6),\ (2,\ 6,\ 3),\ (3,\ 6,\ 2),\ (6,\ 6,\ 1),\ (2,\ 6,\ 4),\ (4,\ 6,\ 2)$

OR 16 possible ways OR $\frac{16}{216}$

adds their probabilities ***(M1)***

Note: Award ***(M1)*** for adding at least 3 of their probabilities (denominator 216).

$(p =)\ \frac{3}{216} + \frac{5}{216} + \frac{14}{216} + \frac{16}{216}$

$(= 0.013889\ldots + 0.023148\ldots + 0.064815\ldots + 0.074074\ldots)$

$= \frac{38}{216}\ \left(= \frac{19}{108},\ \ = 0.176\right)$ ***A1***

METHOD 3

varies b^2 OR b and determines possible values of ac such that $\Delta < 0$ ***(M1)***

correctly determines two of the following six cases ***(A1)***

correctly determines a further two of the following six cases ***(A1)***

correctly determines the remaining two cases ***(A1)***

case 1: $b^2 = 1\ (b = 1)$ 36 possible ways OR $\frac{36}{216}$

case 2: $b^2 = 4\ (b = 2)$ 35 possible ways OR $\frac{35}{216}$

case 3: $b^2 = 9\ (b = 3)$ 33 possible ways OR $\frac{33}{216}$

case 4: $b^2 = 16\ (b = 4)$ 28 possible ways OR $\frac{28}{216}$

case 5: $b^2 = 25\ (b = 5)$ 22 possible ways OR $\frac{22}{216}$

case 6: $b^2 = 36\ (b = 6)$ 19 possible ways OR $\frac{19}{216}$

$(p =)\ 1 - \left(\frac{36}{216} + \frac{35}{216} + \frac{33}{216} + \frac{28}{216} + \frac{22}{216} + \frac{19}{216} + \frac{5}{216}\right)$ ***(M1)***

$(= 1 - (0.16666\ldots + 0.16203\ldots + 0.15277\ldots + 0.12962\ldots + 0.10185\ldots + 0.087962\ldots + 0.023148\ldots))$

Note: Award ***(M1)*** for adding at least 3 of their probabilities inside the above bracket (denominator 216).

$= \frac{38}{216}\ \left(= \frac{19}{108},\ \ = 0.176\right)$ ***A1***

[6 marks]

In parts (g) and (h), consider a randomly generated quadratic function, $f(x) = x^2 + 2Zx + 1$, where the continuous random variable Z~$\mathrm{N}(0,\ 1)$.

(g) Find the probability that the graph of f has two x-intercepts. [3]

Markscheme

recognizes that $4Z^2 - 4 > 0$ $\left(Z^2 > 1\right)$ ***(M1)***

probability of two x-intercepts is

EITHER

$\mathrm{P}(|Z| > 1)$ ***(A1)***

OR

$\mathrm{P}(Z < -1)$ or $\mathrm{P}(Z > 1)$ (can be shown on a labelled diagram) ***(A1)***

$= 0.158655\ldots + 0.158655\ldots$

OR

$1 - \mathrm{P}(-1 \leq Z \leq 1)$ (can be shown on a labelled diagram) ***(A1)***

$= 1 - 0.682689\ldots$

THEN

$= 0.317310\ldots$

$= 0.317$ ***A1***

[3 marks]

The continuous random variables, X_1 and X_2, represent the x-intercepts of the graph of f where $X_1 = -Z - \sqrt{Z^2 - 1}$ and $X_2 = -Z + \sqrt{Z^2 - 1}$.

(h) Given that the graph of f has two x-intercepts, X_1 and X_2, find the probability that both X_1 and X_2 are greater than 0.5. [7]

Markscheme

attempts to solve $X_1 > 0.5$ for Z ***(M1)***

$-1.25 < Z \leq -1$ ***(A1)(A1)***

Note: Award ***(M1)(A1)*** for obtaining $Z = -1.25$ from solving $X_1 = 0.5$ and award ***(A1)*** for stating the correct inequality.

Award ***(M1)(A1)(A1)*** for $-1.25 < Z < -1$.

Award ***(M1)(A1)(A0)*** for $-1.25 < Z$.

Award ***(M1)*** for rearranging to form $-\sqrt{Z^2 - 1} = Z + 0.5$ and then attempting to square both sides $Z^2 - 1 = (Z + 0.5)^2$ $\left(= Z^2 + Z + 0.25\right)$.

attempts to calculate their $\mathrm{P}(X_1,\ X_2 \text{ both } > 0.5)$ ***(M1)***

$\mathrm{P}(-1.25 < Z \leq -1) = 0.053005\ldots$ ***(A1)***

attempts to calculate their $\mathrm{P}(X_1,\ X_2 \text{ both } > 0.5 | x\text{-intercepts})$ ***(M1)***

$= \frac{0.053005\ldots}{0.317310\ldots}$

$= 0.167$ ***A1***

[7 marks]

25. [Maximum mark: 24] 23N.3.AHL.TZ0.1

This question asks you to explore some properties of the family of curves $y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

(a) By systematically varying the value of b, or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x-axis intercepts. [2]

Markscheme

varies the value of b with $a = 3$ ***(M1)***

Note: The ***(M1)*** in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii). Award ***(M1)*** for evidence that $b = 0$ case is considered/determined.

$b = -4,\ 0$ ***A1***

[2 marks]

(b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly

(b.i) one x-axis intercept; [1]

Markscheme

$b < -4$ or $b > 0$ ***A1***

[1 mark]

(b.ii) three x-axis intercepts. [1]

Markscheme

$-4 < b < 0$ ***A1***

[1 mark]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

(c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly

(c.i) two x-axis intercepts; [1]

Markscheme

$b = 0, 4$ *A1*

[1 mark]

(c.ii) one x-axis intercept; [1]

Markscheme

$b < 0$ or $b > 4$ *A1*

[1 mark]

(c.iii) three x-axis intercepts. [1]

Markscheme

$0 < b < 4$ *A1*

[1 mark]

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}, a \neq 0$ and $b \in \mathbb{R}$.

(d) Consider the case where the curve has exactly three x-axis intercepts. State whether each point of zero gradient is located above or below the x-axis. [1]

Markscheme

one point of zero gradient is located on either side (of the x-axis) (or equivalent) *A1*

[1 mark]

(e) Show that the curve has a point of zero gradient at $\mathrm{P}(0,\ b)$ and a point of zero gradient at $\mathrm{Q}\left(-\frac{2}{3}a,\ \frac{4}{27}a^3 + b\right)$. [5]

Markscheme

METHOD 1

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax$ *(A1)*

attempts to solve their $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x *M1*

$x(3x + 2a)(= 0)$ OR $x = \frac{-2a \pm \sqrt{4a^2}}{6}$ OR $x + \frac{a}{3} = \pm\frac{a}{3}$

$x = -\frac{2}{3}a,\ 0$ *A1*

when $x = 0, y = b$ and so $\mathrm{P}(0,\ b)$ is a point of zero gradient *AG*

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ *(M1)*

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b\left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ *A1* so $\mathrm{Q}\left(-\frac{2}{3}a + \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

METHOD 2

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax$ ***(A1)***

substitutes either $x = 0$ or $x = -\frac{2}{3}a$ into their $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

when $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $y = b$ so $\mathrm{P}(0,\ b)$ is a point of zero gradient ***AG***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right)$

$= \frac{4}{3}a^2 - \frac{4}{3}a^2\ (= 0)\ \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2\ (= 0), = \frac{12}{9}a^2 - \frac{4}{3}a^2\ (= 0)\right)$ ***A1***

and so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = -\frac{2}{3}a$ ***AG***

Note: The following two marks are independent of the first three marks.

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b\left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $\mathrm{Q}\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

(f) Consider the points P and Q for $a > 0$ and $b > 0$.

(f.i) Find an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]

Markscheme

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 2a$ ***A1***

when $x = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a\ (a > 0)$ and so (P) is a (local) minimum (point) ***R1***

when $x = -\frac{2}{3}a$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2a\ (a > 0)$ and so (Q) is a (local) maximum (point) ***R1***

[3 marks]

(f.ii) Determine whether each point is located above or below the x-axis. [1]

Markscheme

(P and Q are) both above (the x-axis) ***A1***

Note: Award ***A1*** if it is made clear that both points are above (the x-axis). Accept a labelled sketch that clearly shows this information.

[1 mark]

(g) Consider the points P and Q for $a < 0$ and $b > 0$.

(g.i) State whether P is a local maximum or a local minimum and whether it is above or below the x-axis. [1]

Markscheme

(P) is a (local) maximum (point) and is above (the x-axis) ***A1***

[1 mark]

(g.ii) State the conditions on a and b that determine when Q is below the x-axis. [1]

Markscheme

(Q is below the x-axis when) $\frac{4}{27}a^3 + b < 0$ ***A1***

Note: Award ***A1*** for an equivalent correct inequality, e.g. $\frac{4}{27}a^3 < -b$.

Accept a labelled sketch that clearly shows this information.

[1 mark]

(h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x-axis intercepts. [5]

Markscheme

METHOD 1

attempts to factorize $4a^3b + 27b^2\ (< 0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3 + b\right)\ (< 0)$ OR $b\left(4a^3 + 27b\right)\ (< 0)$ ***A1***

$b > 0$ and $\frac{4}{27}a^3 + b < 0$ OR $b < 0$ and $\frac{4}{27}a^3 + b > 0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.

Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

when b and $\frac{4}{27}a^3 + b$ have opposite sign, P and Q are located on either side (of the x-axis) (or equivalent) ***R1***

Note: Accept labelled sketches that clearly show this information.

P and Q are located on either side (of the x-axis) if (and only if) the curve has exactly three x-axis intercepts ***R1***

if $4a^3b + 27b^2 < 0$, the the graph of $y = x^3 + ax^2 + b$ has exactly three x-axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

METHOD 2

attempts to factorize $4a^3b + 27b^2\ (< 0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3 + b\right)\ (< 0)$ OR $b\left(4a^3 + 27b\right)\ (< 0)$ ***A1***

either $b > 0$ and $\frac{4}{27}a^3 + b < 0$ OR $b < 0$ and $\frac{4}{27}a^3 + b > 0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.

Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

$b > 0$ and $\frac{4}{27}a^3 + b < 0$, $\left(\Rightarrow 0 < b < -\frac{4}{27}a^3\right) \Rightarrow a < 0$ and hence three x-axis intercepts ***R1***

$b < 0$ and $\frac{4}{27}a^3 + b > 0$, $\left(\Rightarrow -\frac{4}{27}a^3 < b < 0\right) \Rightarrow a > 0$ and hence three x-axis intercepts ***R1***

Note: Accept labelled sketches that clearly show this information.

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x-axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2^{nd} ***A1*** line not necessary in reverse method).

[5 marks]

26. [Maximum mark: 31] 23N.3.AHL.TZ0.2

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α.

(a) Consider a family of straight lines, L, with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C, at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

(a.i) Write down an expression for the gradient of L in terms of x and y. [1]

Markscheme

$\frac{y}{x}$ ***A1***

[1 mark]

(a.ii) Hence show that the gradient of C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. [1]

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(\frac{y}{x}\right)} \left(= -\frac{1}{m}\right)$ ***A1***

Note: Award ***A1*** for responses such as the gradient is the negative (opposite) reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).

Award ***A1*** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.

Do not award ***FT*** from part (a) (i).

so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***AG***

[1 mark]

(a.iii) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, show that the family of curves, C, has equation $x^2 + y^2 = k$ where k is a parameter. [2]

Markscheme

attempts to separate variables x and y ***(M1)***

$\int y\,\mathrm{d}y = -\int x\,\mathrm{d}x$

Note: Award *(M1)* for $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

$\frac{y^2}{2} = -\frac{x^2}{2} + c$ ***A1***

Note: Award ***A1*** for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.

Award ***A0*** $\frac{y^2}{2} = -\frac{x^2}{2}$.

$\frac{x^2}{2} + \frac{y^2}{2} = c$

$\Rightarrow x^2 + y^2 = k$ (where $k = 2c$) ***AG***

[2 marks]

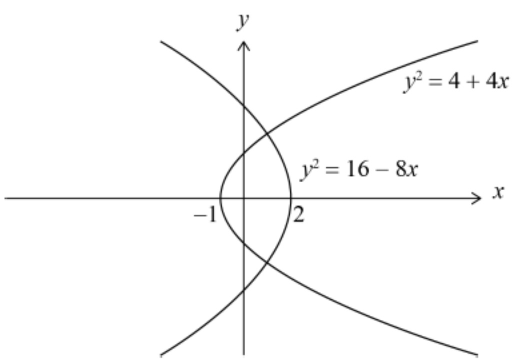
A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

(b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x-intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

Markscheme



two parabolic shaped curves with approximately correct shape/position (e.g. two intersection points, in first and fourth quadrant) ***A1A1***

x-intercepts -1 and 2. ***A1***

[3 marks]

(c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $\mathrm{M}\left(a-b,\ 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b,\ -2\sqrt{ab}\right)$. [6]

Markscheme

at intersection, $4a^2 - 4ax = 4b^2 - 4bx$

$4a^2 - 4b^2 = 4ax + 4bx\ \left(a^2 - b^2 = ax + bx,\ 4a^2 - 4b^2 - 4ax - 4bx = 0\right)$ ***A1*** attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of $a^2 - b^2 - ax - bx = 0$ (or equivalent) ***(M1)***

$(a+b)(a-b) = (a+b)x$

Note: Accept alternative forms such as $4(a+b)(a-b) = 4(a+b)x$ or $(a+b)((a-b)-x) = 0$.

recognition that $a+b > 0$ (or equivalent, e.g. $a > 0, b > 0$) (allows division by $a+b$) ***R1***

Note: Subsequent marks are not dependent on this ***R1***.

$x = a - b$ ***A1***

Note: As $x = a - b$ is an ***AG***, only award the above ***A1*** if $a^2 - b^2 = (a+b)(a-b)$ has been used.

substitutes $x = a - b$ into either $y^2 = 4a^2 - 4ax$ or $y^2 = 4b^2 + 4bx$ and attempts to simplify ***(M1)***

$y^2 = 4a^2 - 4a(a-b)$ OR $y^2 = 4b^2 + 4b(a-b)$

$y^2 = 4a^2 - 4a^2 + 4ab \Rightarrow y = \pm 2\sqrt{ab}$ ***A1***

so $\mathrm{M}\left(a-b,\ 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b,\ -2\sqrt{ab}\right)$ ***AG***

[6 marks]

(d) At point M, show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

Markscheme

METHOD 1

attempts implicit differentiation on either curve ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a}{2y}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4b}{2y}$ (or equivalent) ***A1***

substitutes $y = 2\sqrt{ab}$ into either $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a}{2y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4b}{2y}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\frac{a}{b}}\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{b}{a}}\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

OR

e.g. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

METHOD 2

attempts chain rule differentiation on either $y=\sqrt{4a^2-4ax}$ or $y=\sqrt{4b^2+4bx}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ (or equivalent) ***A1***

substitutes $x=a-b$ into either $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}y}=-\sqrt{\frac{a}{b}}\left(=-\frac{a}{\sqrt{ab}}\right)$ and $\frac{\mathrm{d}x}{\mathrm{d}y}=\sqrt{\frac{b}{a}}\left(=\frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}}\times\sqrt{\frac{b}{a}}=-1$ OR $-\frac{a}{\sqrt{ab}}\times\frac{b}{\sqrt{ab}}=-1$ (or equivalent) ***A1***

OR

e.g. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

[5 marks]

Consider two families of curves, F and G.

The gradient of F is denoted by $f(x,\ y)$.

The gradient of G is denoted by $g(x,\ y)$.

Each member of F intersects every member of G at an acute angle, α.

It can be shown that

$g(x,\ y)=\frac{f(x,\ y)+\tan\alpha}{1-f(x,\ y)\tan\alpha}$.

In part (e), consider the specific case where $f(x,\ y)=-\frac{x}{y}$, for $x\neq0$, $y\neq0$ and $\alpha=\frac{\pi}{4}$.

(e.i) Show that $g(x,\ y)=\frac{y-x}{y+x}$. [2]

Markscheme

$g(x,\ y)=\frac{-\frac{x}{y}+\tan\frac{\pi}{4}}{1-\left(-\frac{x}{y}\right)\tan\frac{\pi}{4}}$ ***(A1)***

$g(x,\ y)=\frac{-\frac{x}{y}+1}{1+\frac{x}{y}}\left(=\frac{\frac{-x+y}{y}}{\frac{y+x}{y}}\right)$ ***A1***

so $g(x,\ y)=\frac{y-x}{y+x}$ ***AG***

[2 marks]

(e.ii) Hence, by solving the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-x}{y+x}$, find a general equation that represents this family of curves, G. Give your answer in the form $h(x,\ y)=d$ where d is a parameter. [9]

Markscheme

let $y = vx$ *(M1)*

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ *(A1)*

$\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x} =\right)\frac{vx-x}{vx+x}\ \left(= \frac{v-1}{v+1}\right)$ *(A1)*

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single fraction in v *(M1)*

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2+1}{v+1}$ (or equivalent) *(A1)*

attempts to separate variables x and v *(M1)*

$\int \frac{v+1}{v^2+1}\mathrm{d}v = -\int \frac{1}{x}\mathrm{d}x$ (or equivalent)

$\frac{1}{2}\ln\left(v^2+1\right) + \arctan\ v = -\ln|x|(+d)$ (or equivalent) ***A1A1***

$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\frac{y}{x} + \ln|x| = d$ (or equivalent) ***A1***

[9 marks]

(f) By considering $\lim\limits_{\alpha\to\frac{\pi}{2}} \tan\ \alpha$, show that, for all finite $f(x,\ y)$,

$\lim\limits_{\alpha\to\frac{\pi}{2}} g(x,\ y) = -\frac{1}{f(x,\ y)}$. [2]

Markscheme

METHOD 1

$g(x,\ y) = \frac{\frac{1}{\tan\ \alpha}f(x,\ y)+1}{\frac{1}{\tan\ \alpha}-f(x,\ y)}$ ***M1***

EITHER

as $\alpha\to\frac{\pi}{2},\ \frac{1}{\tan\ \alpha}\to 0$, (hence $g(x,\ y)\to -\frac{1}{f(x,\ y)}$) ***R1***

OR

as $\alpha\to\frac{\pi}{2},\ \tan\ \alpha\to\infty$ and so $g(x,\ y)\to\frac{0\times f(x,\ y)+1}{0-f(x,\ y)}$ ***R1***

THEN

$\lim\limits_{\alpha\to\frac{\pi}{2}} g(x,\ y) = -\frac{1}{f(x,\ y)}$ ***AG***

Note: The ***R1*** is dependent on the ***M1.***

METHOD 2

uses either $\tan\ \alpha = \frac{\sin\ \alpha}{\cos\ \alpha}$ or $\frac{1}{\tan\ \alpha} = \frac{\cos\ \alpha}{\tan\ \alpha}$ to form $g(x,\ y) = \frac{\cos\ \alpha\ f\ (x,\ y)+\sin\ \alpha}{\cos\ \alpha-\sin\ \alpha\ f\ (x,\ y)}$ ***M1***

as $\alpha\to\frac{\pi}{2},\ \cos\ \alpha\to 0$ and $\sin\ \alpha\to 1$ and so $g(x,\ y)\to\frac{0\times f(x,\ y)+1}{0-f(x,\ y)}$ ***R1***

$\lim_{\alpha \to \frac{\pi}{2}} g\left(x,\ y\right) = -\frac{1}{f(x,\ y)}$ ***AG***

Note: The **R1** is dependent on the **M1.**

[2 marks]

27. [Maximum mark: 25] 23M.3.AHL.TZ1.1

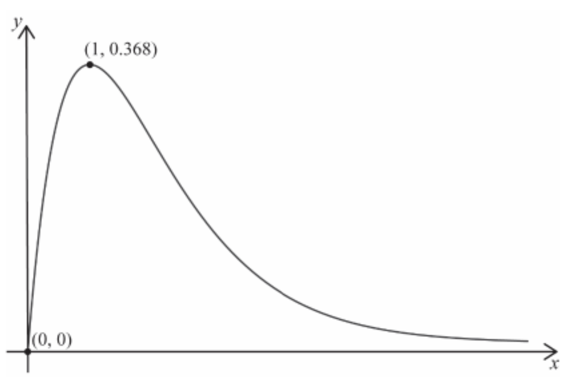
In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = x e^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

Markscheme



A1 for $(1,\ 0.368)$ or $\left(1,\ \frac{1}{e}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through, the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x-axis and the line $x = b$, where $b > 0$ is given by $\frac{e^b - b - 1}{e^b}$. [6]

Markscheme

$\int_0^b x e^{-x} dx$ ***(A1)***

Note: Award ***(A1)*** for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$= [-x\mathrm{e}^{-x}]_0^b + \int_0^b \mathrm{e}^{-x}\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for each part (including the correct sign with each)

$= [-x\mathrm{e}^{-x}]_0^b - [-\mathrm{e}^{-x}]_0^b$ ***A1A1***

Note: Award ***A1*** for correct second term. Condone absence of limits to this point

attempt to substitute limits ***M1***

$= -b\mathrm{e}^{-b} - \mathrm{e}^{-b} + 1$ ***A1***

$= \frac{\mathrm{e}^b - b - 1}{\mathrm{e}^b}$ ***AG***

[6 marks]

You may assume that the total area, A_n, of the region between the graph $y = f_n(x)$ and the x-axis can be written as $A_n = \int_0^\infty f_n(x)\mathrm{d}x$ and is given by $\lim\limits_{b\to\infty} \int_0^b f_n(x)\ \mathrm{d}\ x$.

(c.i) Use l'Hôpital's rule to find $\lim\limits_{b\to\infty} \frac{\mathrm{e}^b - b - 1}{\mathrm{e}^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

Markscheme

$\lim\limits_{b\to\infty} \frac{\mathrm{e}^b - b - 1}{\mathrm{e}^b} = \lim\limits_{b\to\infty} \frac{\mathrm{e}^b - 1}{\mathrm{e}^b}$ ***A1***

Note: Award ***A1*** for correct quotient. Condone absence of limit.

$\left(= \lim\limits_{b\to\infty} \frac{\mathrm{e}^b}{\mathrm{e}^b}\right) = 1$ ***A1***

[2 marks]

(c.ii) Hence write down the value of A_1. [1]

Markscheme

$\left(\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x =\right) 1$ ***A1***

[1 mark]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(d.i) A_4; [2]

Markscheme

correct integral *(M1)*

Note: Award ***M1*** for correct integrand with limits from 0 to a larger number.

24 ***A1***

[2 marks]

(d.ii) A_5. [1]

Markscheme

120 ***A1***

Note: The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) Suggest an expression for A_n in terms of n, where $n \in \mathbb{Z}^+$. [1]

Markscheme

$A_n = n!$ ***A1***

[1 mark]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m, $\lim\limits_{b \to \infty} x^m \mathrm{e}^{-x} = 0$. [8]

Markscheme

Note: Accept starting at $n = 0$, throughout this proof.

$n=1$

$A_1=1$ and $1!=1$ ***M1A1***

Note: Award ***M1*** for considering the case where $n=1$, and ***A1*** if it is clear that both $A_1=1$ and $1!=1$ have been considered.

so true for $n=1$

assume true for $n=k$, ($A_k=\int_0^\infty x^k\mathrm{e}^{-x}\mathrm{d}x=k!$) ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

when $n=k+1$

attempt to integrate by parts ***M1***

Note: To obtain the ***M1***, a minimum of an expression +/- an integral must be seen.

$\int_0^\infty x^{k+1}\mathrm{e}^{-x}\mathrm{d}x=\left[-x^{k+1}\mathrm{e}^{-x}\right]_0^\infty+\left(k+1\right)\int_0^\infty x^k\mathrm{e}^{-x}\mathrm{d}x$ ***A1***

$\left(k+1\right)\int_0^\infty x^k\mathrm{e}^{-x}\mathrm{d}x$ simplified to $(k+1)k!$ seen ***A1***

$=0(k+1)k!$

Note: Condone omission of the zero.

$=(k+1)!$ ***A1***

Hence if true for $n=k$ then also true for $n=k+1$. As true for $n=1$ so true for $n\in\mathbb{Z}^+$. ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[8 marks]

28. [Maximum mark: 30] 23M.3.AHL.TZ1.2

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers $x_1, x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of x_1 and x_2 as a function, f, of x_1 only. [2]

Markscheme

$x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b.i) Find the value of x_1 for which the function is maximum. [1]

Markscheme

$(x_1 =)\ 6$ ***A1***

Note: Award the **A1** if 6 seen in part (ii).

[1 mark]

(b.ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Markscheme

$f(6)$ **OR** 6^2 **OR** graph with maximum at $(6,\ 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S, where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2\left(S\right) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2\left(S\right) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Markscheme

$M_2\left(12\right) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

Consider n positive real numbers, $x_1, x_2, \ldots, x_n$.

The geometric mean is defined as $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1+x_2+\ldots+x_n)}{n}$.

(d.i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \ldots = x_n$. [2]

Markscheme

let all x_1 be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(d.ii) Use this result to prove that $M_n\left(S\right) = \left(\frac{S}{n}\right)^n$. [4]

Markscheme

$x_1 + x_2 + \ldots + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \leq \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times \ldots \times x_n \leq \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n\left(S\right) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) Hence determine the value of

(e.i) $M_3(12)$; [1]

Markscheme

$M_3\left(12\right) = 4^3 = 64$ ***A1***

[1 mark]

(e.ii) $M_4(12)$; [1]

Markscheme

$M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(e.iii) $M_5(12)$. [1]

Markscheme

$M_5(12) = 2.4^5 = 79.6\ (79.6262\ldots)$ ***A1***

[1 mark]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n.

(f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]

Markscheme

considering $M_n(12)$ for higher values of n

$P(12) = 81$ ***A1***

$n = 4$ ***A1***

Note: Award ***A0A0*** for $P(12) = 82.6$ and $n = 4.41$.

[2 marks]

(g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

Markscheme

Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n = 7$ ***A1***

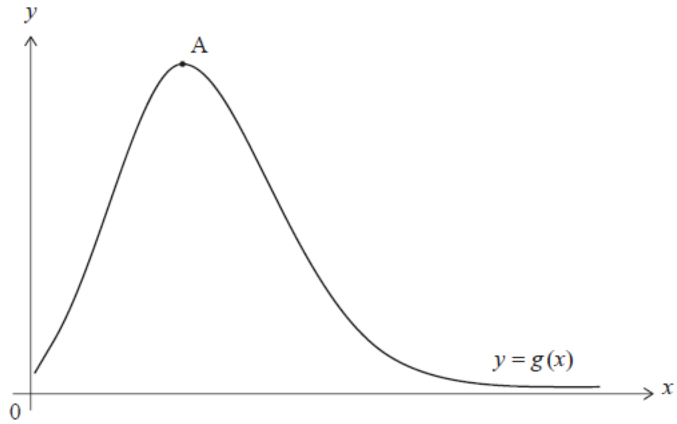
$P(20) = \left(\frac{20}{7}\right)^7 = 1550\ (1554.260\ldots)$ ***A1***

Note: Award *(M1)A0A1* for $n = 7.36$ and $P(20) = 1570$.

[3 marks]

Consider the function g, defined by $\ln\left(g(x)\right) = x\ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



(h) Find, in terms of S, the x-coordinate of point A. [6]

Markscheme

EITHER

$\ln(g(x)) = x(\ln(S) - \ln x)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g\prime(x)}{g(x)} = \ln S - \ln x - x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g\prime(x)}{g(x)} = \ln\frac{S}{x} + \left(x\frac{x}{S} \times \frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g\left(x\right) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g\prime\left(x\right) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x \times \frac{x}{S} \times \left(-Sx^{-2}\right) + \ln\left(\frac{S}{x}\right)\right]$

$= \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right) - 1\right]$ ***A1***

THEN

$g\prime(x) = \left(\ln \frac{S}{x} - 1\right) g(x)$

$g(x) \neq 0$

$g\prime(x) = 0 \Rightarrow \ln \frac{S}{x} - 1 = 0$ ***M1***

$x = \frac{S}{\mathrm{e}}\ \left(0.368\,S,\ 0.36789\ldots S\right)$ ***A1***

[6 marks]

(i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]

Markscheme

$\ln\left(g\left(x\right)\right) = x\ln\left(\frac{S}{x}\right) \Rightarrow \ln\left(g\left(x\right)\right) = \ln\left(\frac{S}{x}\right)^x$ ***M1***

$g\left(x\right) = \left(\frac{S}{x}\right)^x$ ***A1***

$= M_x(S)$ for $x \in \mathbb{Z}^+$ ***AG***

[2 marks]

(j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}^+$. [3]

Markscheme

$\frac{100}{\mathrm{e}} = 36.8$ ***M1***

$\left(\frac{100}{36}\right)^{36} = 9.3996\ldots \times 10^{15}$ **AND** $\left(\frac{100}{37}\right)^{37} = 9.47406\ldots \times 10^{15}$ ***R1***

largest possible product is 9.47×10^{15} $\left(9.4706\ldots \times 10^{15}\right)$ ***A1***

Note: Award ***A1*** independently of the ***R1*** (but not independently of the ***M1***).

[3 marks]

29. [Maximum mark: 24] 23M.3.AHL.TZ2.1

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a.

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator "logarithm to any base feature".

The function f is defined by

$f(x) = \log_a x$ where $x \in \mathbb{R}^+$ and $a \in \mathbb{R}^+, a \neq 1$.

(a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

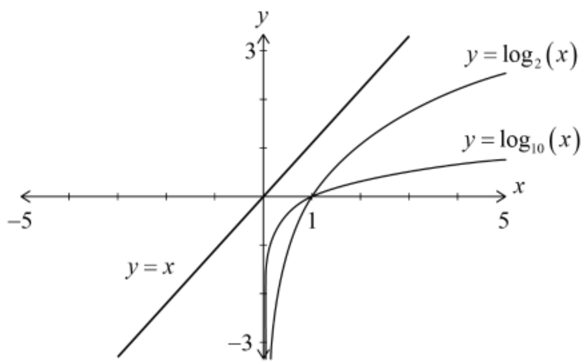
$y = \log_2 x$

$y = \log_{10} x$

$y = x$.

Clearly label each graph with its equation and state the value of any non-zero x-axis intercepts. [4]

Markscheme



clearly labelled graphs of $y = \log_2 x$ and $y = \log_{10} x$ with correct domain, asymptotic behaviour and concavity evident ***A1***

correct relative positions of the two log graphs both above and below the x-axis ***A1***

$(1, 0)$ indicated (coordinates not required) ***A1***

correct graph of $y = x$ ***A1***

[4 marks]

In parts (b) and (c), consider the case where $a = \mathrm{e}$. Note that $\ln x \equiv \log_{\mathrm{e}} x$.

(b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a minimum. [5]

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}(x - \ln x)$

$= 1 - \frac{1}{x}$ ***A1***

attempts to solve their $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x ***(M1)***

$1 - \frac{1}{x} = 0 \Rightarrow x = 1$

(when $x = 1$,) $x - \ln x = 1$ ***A1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-\frac{1}{x}\right)$

$=\frac{1}{x^2}$ ***A1***

$\frac{1}{x^2}>0$ (when $x=1$) ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1*** for either '$1>0$' or a graph of $y=\frac{1}{x^2}>0$ or 'the graph of $y=x-\ln x$ is concave-up'. Do not award ***R1*** if the second derivative is incorrect.

OR

for $(0<)x<1, 1-\frac{1}{x}<0$ ***R1***

for $x>1, 1-\frac{1}{x}>0$ ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1R1*** for either a clearly labelled sign diagram/table (accept correct numerical values) or the graph of $y=1-\frac{1}{x}$ with sign change in gradient indicated.

Note: Award a maximum of ***A0(M1)A1A0R1*** or ***A0(M1)A1R0R1*** if no symbolic derivatives are seen.

[5 marks]

(c) Hence deduce that $x>\ln x$. [1]

Markscheme

EITHER

$x-\ln x\geq 1\,(x\in\mathbb{R}^+)$ ***R1***

OR

$x-\ln x>0\,(x\in\mathbb{R}^+)$ ***R1***

THEN

so $x>\ln x$ ***AG***

[1 mark]

(d) There exist values of a for which the graph of $y=\log_a x$ and the line $y=x$ do have intersection points. The following table gives three intervals for the value of a.

Interval	Number of intersection points

$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a, write down the values of p, q and r. [4]

Markscheme

Interval	Number of intersection points
$0 < a < 1$	$p = 1$
$1 < a < 1.4$	$q = 2$
$1.5 < a < 2$	$r = 0$

A1A2A1

Note: Award ***A1*** for $p = 1$, ***A2*** for $q = 2$ and ***A1*** for $r = 0$.

[4 marks]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

(e) Find the exact coordinates of P and the exact value of a. [8]

Markscheme

METHOD 1

EITHER

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\frac{\ln x}{\ln a} = \frac{1}{\ln a}$ OR $\ln x = 1$ OR $\left(e^{\frac{1}{x}}\right)^x = x$ for x ***(M1)***

$x = e$

coordinates of P are (e, e) (accept $x = e, y = e$) ***A1A1***

attempts to solve $\frac{1}{\ln a} = e$ OR $\log_a e = e$ for a analytically ***(M1)***

$\ln a = \frac{1}{e}$ OR $a^e = e$

$a = e^{\frac{1}{e}}$ ***A1***

METHOD 2

EITHER

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\log_a\left(\frac{1}{\ln a}\right) = \frac{1}{\ln a}$ for a ***(M1)***

EITHER

$\frac{\ln a\left(\frac{1}{\ln a}\right)}{\ln a} = \frac{1}{\ln a} \Rightarrow \ln\left(\frac{1}{\ln a}\right) = 1$

OR

for example, writes $a^{\log_a\left(\frac{1}{\ln a}\right)} = a^{\frac{1}{\ln a}}$ and then attempts to apply appropriate index/log laws to both sides: $\ln a = \frac{\log_a a}{\log_a e}$ and so $\frac{1}{\ln a} = \log_a e$

$a^{\frac{1}{\ln a}} = a^{\log_a e} = e$

THEN

attempts to solve $\frac{1}{\ln a} = \mathrm{e}$ OR $\log_a \mathrm{e} = \mathrm{e}$ for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ OR $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

$x = \frac{1}{\ln \mathrm{e}^{\frac{1}{\mathrm{e}}}} = \frac{1}{\frac{1}{\mathrm{e}}}$

coordinates of P are (e, e) (accept $x = \mathrm{e}, y = \mathrm{e}$) ***A1A1***

METHOD 3

$y = \log_{\mathrm{a}} x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

(equation of the tangent at (x_1, y_1) is) $y = \frac{1}{x_1 \ln a}(x - x_1) + \frac{\ln x_1}{\ln a}$ (or equivalent) ***A1***

compares this equation with $y = x$ and attempts to form at least one of the following ***M1***

$\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$

attempts to solve $\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$ for x_1 ***(M1)***

$x_1 = \mathrm{e}$

coordinates of P are (e, e) (accept $x = \mathrm{e}, y = \mathrm{e}$) ***A1A1***

attempts to solve $\frac{1}{e \ln a} = 1$ (or equivalent) for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ **OR** $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

[8 marks]

(f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have

(f.i) two intersection points; [1]

Markscheme

$1 < a < \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

NOTE: Award ***A0*** for $a < \mathrm{e}^{\frac{1}{\mathrm{e}}}$

[1 mark]

(f.ii) no intersection points. [1]

Markscheme

$a > e^{\frac{1}{e}}$ ***A1***

Note: Only award ***FT*** for $1.4 < a < 1.5$. If the value of a is not exact, e.g. 1.44, award at most ***A0A1*** in part (f) for a consistent approximate endpoint value.

If a value of a is not found in part (e), award at most ***A0A1*** in part (f) for a consistent approximate endpoint value provided that $1.4 < a < 1.5$.

[1 mark]

30. [Maximum mark: 31] 23M.3.AHL.TZ2.2

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where $m, c \in \mathbb{R}$ and $m, c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c, in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

(a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Markscheme

$m = 2, c = -1$

$r = \frac{1}{2}$ ***(A1)***

$2, \frac{1}{2}, -1$

EITHER

$d\left(= \frac{1}{2} - 2 = -1 - \frac{1}{2}\right) = -\frac{3}{2}$ ***A1***

OR

this sequence has a common difference of $-\frac{3}{2}$ ***A1***

OR

the (arithmetic) mean of 2 and -1 is $\frac{1}{2}$ ***A1***

THEN

hence $L(x) = 2x - 1$ is an AS-linear function ***AG***

[2 marks]

Consider $L(x) = mx + c$.

(b.i) Show that $r = -\frac{c}{m}$. [1]

Markscheme

$(L(r) = 0 \Rightarrow) mr + c = 0$ ***A1***

$r = -\frac{c}{m}$ ***AG***

Note: Award ***A0*** for numerical verification from $L(x) = 2x - 1$ in part (a).

[1 mark]

(b.ii) Given that $L(x)$ is an AS-linear function, show that $L\left(x\right) = mx - \frac{m^2}{m+2}$. [4]

Markscheme

METHOD 1

EITHER

attempts to use $(d =) r - m = c - r$ ***(M1)***

Note: Award ***(M1)*** for attempting to use $(d =) m - r = r - c$.

$\left(d =\right) - \frac{c}{m} - m = c - \left(-\frac{c}{m}\right)$ ***A1***

Note: Award ***A1*** for $\left(d =\right) - \frac{c}{m} - m = \frac{c-m}{2}$.

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $\frac{m+c}{2} = r$ ***(M1)***

$m + c = -\frac{2c}{m}$ ***A1***

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $c = m + 2d$ ***(M1)***

$c = m + 2\left(-\frac{c}{m} - m\right)$ ***A1***

Note: Award ***A1*** for $c = m + 2\left(c - \left(-\frac{c}{m}\right)\right)$.

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $r = m + d$ and $c = m + 2d$ $(c = m + 2(r - m))$ ***(M1)***

$m^2 + dm + m + 2d = 0$ (or equivalent) ***A1***

substitutes $d = \frac{c-m}{2}$ into their expression involving m and d ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

THEN

$c\left(m + 2\right) = -m^2 \Rightarrow c = -\frac{m^2}{m+2}$ ***A1***

Note: Award ***A1*** for a convincing demonstration that $c = -\frac{m^2}{m+2}$.

so $L\left(x\right) = mx - \frac{m^2}{m+2}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

METHOD 2

considers $L(x) = mx - mr$

attempts to use $(d =)r - m = c - r$ ***(M1)***

Note: Award ***(M1)*** for attempting to use $(d =)m - r = r - c$.

$(d =)r - m = -mr - r$ ***A1***

attempts to express r in terms of m ***(M1)***

$2r + mr = m \Rightarrow r = \frac{m}{m+2}$ ***A1***

so $L\left(x\right) = mx - \frac{m^2}{m+2}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

[4 marks]

(b.iii) State any further restrictions on the value of m. [1]

Markscheme

$m \neq -2 \ (m \neq 0)$ ***A1***

[1 mark]

There are only **three** integer sets of values of m, r and c, that form an AS-linear function. One of these is $L(x) = -x - 1$.

(c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c. [3]

Markscheme

attempts to find an integer value of m ***(M1)***

e.g. uses the result that $m + 2$ exactly divides 2 OR uses a table OR uses a graph and slider OR uses systematic trial and error

Note: Award ***(M1)*** for solving $m^2 = k\left(m + 2\right)$ for m or solving $mr - \frac{m^2}{m+2} = 0$ for m or solving $m^2 + cm + 2c = 0$ for m.

$m = -4$ OR $m = -3$ ***(A1)***

$-4,\ 2,\ 8$ OR $-3,\ 3,\ 9$

$L(x) = -4x + 8,\ L(x) = -3x + 9$ ***A1***

Note: Award ***(M1)(A1)A0*** for $-4x + 8$ and $-3x + 9$.

[3 marks]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}, a \neq 0$ and $b, c \in \mathbb{R}$.

Let $r_1, r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

(d) Write down an expression for

(d.i) the sum of roots, $r_1 + r_2$, in terms of a and b. [1]

Markscheme

$-\frac{b}{a}$ ***A1***

[1 mark]

(d.ii) the product of roots, $r_1 r_2$, in terms of a and c. [1]

Markscheme

$\frac{c}{a}$ ***A1***

[1 mark]

If a, r_1, b, r_2 and c, in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(e.i) write down an expression for $r_2 - r_1$ in terms of a and b; [1]

Markscheme

$b - a$ ***A1***

Note: Award marks as appropriate in parts (e) (ii) and (iii) for use of $r_1 - r_b = a - b$.

[1 mark]

(e.ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \frac{a^2 - ab - b}{2a}$. [2]

Markscheme

attempts to eliminate r_2 ***M1***

$2r_1 = -\frac{b}{a} - \left(b - a\right) \Rightarrow 2r_1 = \frac{a^2 - ab - b}{a}$ (or equivalent) ***A1***

Note: Award ***A1*** for a correct alternative form of $\pm r_1$ or $\pm 2r_1$.

so $r_1 = \frac{a^2 - ab - b}{2a}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

[2 marks]

(e.iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\frac{1}{2}$. [3]

Markscheme

METHOD 1

EITHER

$\left(r_1 =\right) \frac{a+b}{2}$ ***(A1)***

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and b ***M1***

$\frac{a+b}{2} = \frac{a^2 - ab - b}{2a}$ OR $a + b = \frac{a^2 - ab - b}{a}$

OR

$b - r_1 = r_1 - a$ ***(A1)***

attempts to use $b - r_1 = r_1 - a$ with $r_1 = \frac{a^2 - ab - b}{2a}$ ***M1***

$b - \left(\frac{a^2 - ab - b}{2a}\right) = \frac{a^2 - ab - b}{2a} - a$

OR

$(r_1 =) a + d$ ***(A1)***

attempts to use $r_1 = a + d$ with $r_1 = \frac{a^2 - ab - b}{2a}$ and $d = \frac{b-a}{2}$ ***M1***

$\frac{a^2 - ab - b}{2a} = a + \frac{b-a}{2}$

THEN

$2a^2 + 2ab = 2a^2 - 2ab - 2b$ OR $a^2 + ab = a^2 - ab - b$

$4ab + 2b = 0$ OR $2ab + b = 0$

$2b(2a + 1) = 0$ OR $b(2a + 1) = 0$ ***A1***

Note: Award ***(A1)M1*** for any valid approach that correctly leads to $2ab + b = 0$ (or equivalent).
Do not accept numerical verification from the ***AG***.

so $b = 0$ or $a = -\frac{1}{2}$ ***AG***

METHOD 2

$(b =)a + 2d$ OR $(r_1 =)a + d$ ***(A1)***

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and d ***M1***

$a + d = \frac{a^2 - a(a+2d) - (a+2d)}{2a}$ **OR** $2\left(a + d\right) = \frac{a^2 - a(a+2d) - (a+2d)}{a}$

$2a^2 + 4ad + a + 2d = 0$

$(2a + 1)(a + 2d) = 0$ ***A1***

Note: Do not accept numerical verification from the ***AG***.

so $b = 0$ or $a = -\frac{1}{2}$ ***AG***

[3 marks]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Markscheme

METHOD 1

$r_1 = \frac{a}{2}$ OR $r_2 = -\frac{a}{2}$ OR $d = -\frac{a}{2}$ ***(A1)***

$c = -a$ ***(A1)***

attempts to find the values of a ***(M1)***

EITHER

the roots of $ax^2 - a = 0$ are ± 1 and $\frac{a}{2} = \pm 1$

OR

substitutes $x = \pm\frac{a}{2}$ into $ax^2 - a = 0$ giving $\frac{a^3}{4} - a = 0$

OR

$(r_1r_2 =)\frac{c}{a} = -\frac{a^2}{4} \Rightarrow c = \frac{a^3}{4}$ and so $-a = -\frac{a^3}{4} \Rightarrow \frac{a^3}{4} - a = 0$

Note: Award ***(M1)*** for attempting to find the values of a from their arithmetic sequence expressed in terms of a.

OR

$c - r_2 = r_2 - b \Rightarrow -\frac{a^3}{4} - \left(-\frac{a}{2}\right) = -\frac{a}{2} \Rightarrow \frac{a^3}{4} - a = 0$

THEN

$a = \pm 2$ ***(A1)***

$(r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2,\ Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $a = 2$ or $a = -2$.

METHOD 2

$r_1 = -d$ OR $r_2 = d$ OR $a = -2d$ ***(A1)***

$c = 2d$ ***(A1)***

attempts to find the values of d ***(M1)***

EITHER

the roots of $-2dx^2 + 2d = 0$ are ± 1

OR

substitutes $x = \pm d$ into $-2dx^2 + 2d = 0$ giving $-2d^3 + 2d = 0$

OR

attempts to use $r_1r_2 = \frac{c}{a}$ to form $-d^2 = \frac{2d}{-2d}$

Note: Award ***(M1)*** for attempting to find the values of d from their arithmetic sequence expressed in terms of d.

THEN

$d = \pm 1$ ***(A1)***

$(a = \pm 2,\ r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2,\ Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $d = 1$ or $d = -1$.

METHOD 3

$a = 2r_1$ OR $r_2 = -r_1$ OR $d = -r_1$ ***(A1)***

$c = -2r_1$ ***(A1)***

attempts to find the values of r_1 ***(M1)***

EITHER

the roots of $2r_1x^2 - 2r_1 = 0$ are ± 1

OR

substitutes $x = \pm r_1$ into $2r_1x^2 - 2r_1 = 0$ giving $2{r_1}^3 - 2r_1 = 0$

OR

attempts to use $r_1 r_2 = \frac{c}{a}$ to form $-{r_1}^2 = \frac{-2r_1}{2r_1}$

Note: Award ***(M1)*** for attempting to find the values of r_1 from their arithmetic sequence expressed in terms of r_1.

THEN

$r_1 = \pm 1$ ***(A1)***

$(a = \pm 2,\ r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2,\ Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $r_1 = 1$ or $r_1 = -1$.

[5 marks]

Now consider the case where $a = -\frac{1}{2}$.

(g.i) Find an expression for r_1 in terms of b. [2]

Markscheme

attempts to express r_1 in terms of b with $a = -\frac{1}{2}$. ***(M1)***

Note: Do not award ***(M1)*** if $a = \frac{1}{2}$ is used.

EITHER

uses $r_1 = \frac{a+b}{2}$

OR

uses $r_1 = \frac{a^2-ab-b}{2a}$

OR

uses $r_1 - a = b - r_1$

THEN

$r_1 = \frac{2b-1}{4} \left(= \frac{b}{2} - \frac{1}{4}, = \frac{b-\frac{1}{2}}{2}\right)$ ***A1***

[2 marks]

(g.ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\frac{-p \pm q\sqrt{s}}{2}$ where $p, q, s \in \mathbb{Z}^+$. [5]

Markscheme

METHOD 1

EITHER

substitutes their expression for r_1 with $a = -\frac{1}{2}$ into $Q(x)(= 0)$ ***(M1)***

$Q\left(\frac{2b-1}{4}\right)\left(= 0\right) \Rightarrow -\frac{1}{2}\left(\frac{2b-1}{4}\right)^2 + b\left(\frac{2b-1}{4}\right) + c\left(= 0\right)$

OR

$r_2 = \frac{6b+1}{4} \left(= \frac{3b}{2} + \frac{1}{4}\right)$

substitutes their expression for r_2 with $a = -\frac{1}{2}$ into $Q(x)(= 0)$ ***(M1)***

$Q\left(\frac{6b+1}{4}\right)\left(= 0\right) \Rightarrow -\frac{1}{2}\left(\frac{6b+1}{4}\right)^2 + b\left(\frac{6b+1}{4}\right) + c\left(= 0\right)$

THEN

$c = \frac{4b+1}{2}$ $\left(= 2b + \frac{1}{2}\right)$ (seen anywhere) ***A1***

$4b^2 + 20b + 5 = 0$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

METHOD 2

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $Q(x)$ ***(M1)***

$-\frac{1}{2}\left(x - \left(\frac{2b-1}{4}\right)\right)\left(x - \left(\frac{6b+1}{4}\right)\right)$

$-\frac{1}{2}x^2 + bx - \frac{3}{8}b^2 + \frac{1}{8}b + \frac{1}{32}$

$c = \frac{4b+1}{2}$ $\left(= 2b + \frac{1}{2}\right)$ (seen anywhere) ***A1***

$2b + \frac{1}{2} = -\frac{3}{8}b^2 + \frac{1}{8}b + \frac{1}{32}$

$4b^2 + 20b + 5 = 0$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

METHOD 3

$r_2 = \frac{6b+1}{4}$ $\left(= \frac{3b}{2} + \frac{1}{4}\right)$

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ ***(M1)***

$\left(\frac{2b-1}{4}\right)\left(\frac{6b+1}{4}\right) = \frac{c}{-\frac{1}{2}}$ (or equivalent)

EITHER

$c = \frac{4b+1}{2}$ $\left(= 2b + \frac{1}{2}\right)$ (seen anywhere) ***A1***

OR

$-\frac{1}{2}\left(\frac{2b-1}{4}\right)\left(\frac{6b+1}{4}\right)-\left(\frac{6b+1}{4}\right)=\frac{2b-1}{4}-\left(-\frac{1}{2}\right)$ ***A1***

THEN

$4b^2+20b+5=0$

attempts to solve their quadratic in b ***(M1)***

$b=\frac{-5\pm2\sqrt{5}}{2}$ ***A1***

substitutes into $c=\frac{4b+1}{2}$

$c=\frac{-9\pm4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

METHOD 4

attempts to equate two expressions for r_1 with $a=-\frac{1}{2}$ ***(M1)***

$\frac{-b\pm\sqrt{b^2+2c}}{-1}=\frac{2b-1}{4}\left(\pm\sqrt{b^2+2c}=\frac{2b+1}{4}\right)$

$c=\frac{4b+1}{2}\left(=2b+\frac{1}{2}\right)$ (seen anywhere) ***A1***

$12b^2-4b-1+32\left(2b+\frac{1}{2}\right)=0\left(4b^2+20b+5=0\right)$

attempts to solve their quadratic in b ***(M1)***

$b=\frac{-5\pm2\sqrt{5}}{2}$ ***A1***

substitutes into $c=\frac{4b+1}{2}$

$c=\frac{-9\pm4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

METHOD 5

EITHER

$r_1=d-\frac{1}{2}$

substitutes their expression for r_1 in terms of $d$with $a=-\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$Q\left(d-\frac{1}{2}\right)\left(=0\right)\Rightarrow-\frac{1}{2}\left(d-\frac{1}{2}\right)^2+b\left(d-\frac{1}{2}\right)+c\left(=0\right)$

OR

$r_2 = 3d - \frac{1}{2}$

substitutes their expression for r_2 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$Q\left(3d - \frac{1}{2}\right)\left(=0\right) \Rightarrow -\frac{1}{2}\left(3d - \frac{1}{2}\right)^2 + b\left(3d - \frac{1}{2}\right) + c\left(=0\right)$

THEN

$b = 2d - \frac{1}{2}$ and $c = 4d - \frac{1}{2}$ (seen anywhere) ***A1***

$4d^2 + 8d - 1 = 0$

attempts to solve their quadratic in d ***(M1)***

$d = \frac{-2 \pm \sqrt{5}}{2}$

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

METHOD 6

$r_1 = d - \frac{1}{2}$ and $r_2 = 3d - \frac{1}{2}$

substitutes their expression for r_1 and r_2 in terms of d with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ ***(M1)***

$\left(d - \frac{1}{2}\right)\left(3d - \frac{1}{2}\right) = \frac{c}{-\frac{1}{2}}$ (or equivalent)

$c = 4d - \frac{1}{2}$

$4d^2 + 8d - 1 = 0$

attempts to solve their quadratic in d ***(M1)***

$d = \frac{-2 \pm \sqrt{5}}{2}$

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

[5 marks]