CHAPTER SEVEN: SQUARE ROOTS

Test: Tuesday, March 31

Algebraic equations are solved by applying inverse functions. Since quadratics involve squaring values, quadratics can be solved by taking square roots. Every number except zero has two square roots, which are the same as each other except one is positive and one is negative. The square roots of a square number are integers. The square roots of a fraction of squares are rational numbers. The square roots of any other positive number are irrational numbers, meaning they cannot be exactly written as a decimal. The square roots of a negative number are called imaginary numbers, and they are not on the real number line.

7-A Classification of Numbers — Monday • 3/9

natural • integer • rational • irrational • real • complex • imaginary

❶ Identify a number as natural, integer, rational, real, or complex.

7-B Radicals — Tuesday • 3/10

radical • radicand

❶ Add and subtract radical expressions.

❷ Multiply radical expressions.

❸ Simplify the square root of a positive integer.

❹ Simplify the square root of an algebraic term.

❺ Rationalize a denominator.

7-C Complex Numbers — Monday • 3/16

i • real number • imaginary number • complex number • standard form • complex plane • complex conjugate

❶ Plot numbers on the complex plane.

❷ Add and subtract complex numbers.

❸ Simplify the square root of a negative integer.

❹ Multiply complex numbers.

❺ Divide by i.

7-D Completing the Square — Thursday • 3/19

❶ Solve $x^2 + bx + c = 0$ by completing the square.

7-E The Quadratic Formula — Monday • 3/23

quadratic formula • discriminant

❶ Solve a quadratic equation by using the quadratic formula.

❷ Use a discriminant to determine the type of solutions of a quadratic equation.

❸ Explain how zeros, roots, solutions, and x-intercepts are related for a given polynomial.

❹ Use a discriminant to determine the number of x-intercepts of a quadratic function.

7-A Classification of Numbers

Every number is contained in one or more of five number sets.

NATURAL Numbers ($\mathbb{N}$) are counting numbers such as 1, 2, 3, and 2000. They don't require any symbols to write them.

INTEGERS ($\mathbb{Z}$) are the natural numbers and the negative versions of them, and zero.

RATIONAL Numbers ($\mathbb{Q}$) are any integer divided by an integer other than zero.

REAL Numbers ($\mathbb{R}$) are all integers and all numbers inbetween. For example $\sqrt{3}$ is between 1 and 2, and π is between 3 and 4.

COMPLEX Numbers ($\mathbb{C}$) are all numbers, including those involving the square root of a negative such as $\sqrt{-1}$ or $2+\sqrt{-3}$.

Each set is completely included in the sets below it. For example, all rational numbers are also real and complex.

Numbers that are real but not rational are IRRATIONAL.

Numbers that are complex but not real are IMAGINARY.

Subsets of these sets that contain only positive or only negative numbers are denoted with a superscript + or -, such as $\mathbb{R}^+$.

❶ Identify a number as natural, integer, rational, real, or complex.

1. All numbers are complex.
2. If it does not involve an even root of a negative number, it is also real. Otherwise it is imaginary.
3. If it can be written exactly as a fraction of integers, it is also rational. Otherwise it is irrational if it is not imaginary.
   Keep in mind that all decimals can be written as fractions except those with no pattern or end. For example, $0.29 = \frac{29}{100}$ and $0.292929... = \frac{29}{99}$.
4. If it can be written with no decimal point, fraction bar, or any other symbol other than a negative sign, it is also an integer.
5. If it is a positive integer, it is natural.

❶ Sort the following numbers into the smallest set that includes each: 2, -2, $\sqrt{2}$, $\sqrt{-2}$, -22, 2.2, $2-\sqrt{-2}$, 2.2×10^{22}, 2.2×10^{-22}, π.

1. $\sqrt{-2}$ and $2-\sqrt{-2}$ involve square roots of negative numbers. They are complex but not real.
2. $\pi \approx 3.1416$ and $\sqrt{2} \approx 1.4142$ are not imaginary, but they cannot be written as exact decimals. They are real but not rational.
3. 2.2 and 2.2×10^{-22} can be written as exact fractions, but they are not whole numbers. They are rational but not integers.
4. -2 and -22 can be written without fractions, decimals, radicals, etc., but they are not positive. They are integers but not natural.
5. 2 and 2.2×10^{22} are positive whole numbers. They are natural.

7-B Radicals

An expression under a root sign is a RADICAND. Together with the root sign it is called a RADICAL.

Radical expressions can be simplified by combining like terms. Factors multiplied by the same radical, such as $2\sqrt{3}$ and $-9\sqrt{3}$, are like terms.

❶ Add and subtract radical expressions.

1. Distribute any coefficient, including negatives.
2. Combine like terms.

❶ $(3 + 5\sqrt{7}) - 2(3\sqrt{2} - 11\sqrt{7})$

1. $3 + 5\sqrt{7} - 6\sqrt{2} + 22\sqrt{7}$
2. $3 + (5\sqrt{7} + 22\sqrt{7}) - 6\sqrt{2}$

$3 + 27\sqrt{7} - 6\sqrt{2}$

For positive numbers a and b, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

❷ Multiply radical expressions.

1. Distribute.
2. Multiply the radicands together.

❷

a) $\sqrt{6} \cdot \sqrt{3}$

1.

2. $\sqrt{18}$

b) $\sqrt{6}(-3 + \sqrt{3} - 5\sqrt{11})$

$-3\sqrt{6} + \sqrt{6}(\sqrt{3}) - \sqrt{6}(5\sqrt{11})$

$-3\sqrt{6} + \sqrt{18} - 5\sqrt{66}$

Just as $\sqrt{a}$ and $\sqrt{b}$ can be combined (by multiplying) into $\sqrt{ab}$, $\sqrt{ab}$ can be split into $\sqrt{a}\sqrt{b}$. Doing so with one of the values being a square allows that part of the radical to be simplified, such as $\sqrt{12}$ being split into $\sqrt{4}\sqrt{3}$ which is $2\sqrt{3}$.

❸ Simplify the square root of a positive integer.

1. Identify a square number that divides evenly into the radicand.
2. Factor the radicand by dividing by the square number above. Leave each factor in a square root.
3. Take the square root of the square factor, and leave the other factor in the radical.
4. Repeat these steps if there is another square factor in the radical.

❸ $\sqrt{800}$

1. $\sqrt{100}\sqrt{8}$
2. $10\sqrt{8}$
3. $10\sqrt{4}\sqrt{2} = 10(2)\sqrt{2} = 20\sqrt{2}$

A variable to an even power is a square: $x^{2n} = (x^n)^2$. A variable to an odd power is the variable times a square: $x^{2n+1} = x(x^n)^2$.

❹ Simplify the square root of an algebraic term.

1. Do steps 1 and 2, above.
2. Rewrite variables with an odd exponent as the variable multiplied by the variable to an exponent that is lower by one, such as changing x^5 into xx^4.
3. Move all the even exponent factors into the radical with the square number.
4. Do steps 3 and 4, above.

❹ $\sqrt{50a^{20}b^{21}c^{49}d}$

1. $\sqrt{25}\sqrt{2a^{20}b^{21}c^{49}d}$
2. $\sqrt{25}\sqrt{2a^{20}bb^{20}cc^{48}d}$
3. $\sqrt{25a^{20}b^{20}c^{48}}\sqrt{2bcd}$
4. $5a^{10}b^{10}c^{24}\sqrt{2bcd}$

The CONJUGATE of a number $a+\sqrt{b}$ is $a-\sqrt{b}$.

Multiplying a number by its conjugate results in a rational (nonradical) number: $(a+\sqrt{b})\,(a-\sqrt{b})=a^2+a\sqrt{b}-a\sqrt{b}-(\sqrt{b})^2=a^2-b$.

A fraction with a radical in the denominator is not considered simplified. To RATIONALIZE a Denominator is to rewrite a fraction so that there is no radical in the denominator.

❺ Rationalize a denominator.

1. Multiply the numerator and denominator by the denominator if it is a single term, or by the conjugate of the denominator if it has two terms.
2. Simplify the radical.
3. Reduce.

❺ Simplify.

a) $\frac{12}{\sqrt{20}}$

1. $\frac{12}{\sqrt{20}}\cdot\frac{\sqrt{20}}{\sqrt{20}}=\frac{12\sqrt{20}}{20}$
2. $\frac{12(2\sqrt{5})}{20}=\frac{12(2\sqrt{5})}{20}=\frac{24\sqrt{5}}{20}$
3. $\frac{6\sqrt{5}}{5}$

b) $\frac{12}{8-\sqrt{20}}$

$\frac{12}{8-\sqrt{20}}\cdot\frac{8+\sqrt{20}}{8+\sqrt{20}}=\frac{96+12\sqrt{20}}{8^2-20}$

$\frac{96+12(2\sqrt{5})}{44}=\frac{96+24\sqrt{5}}{44}$

$\frac{24+6\sqrt{5}}{11}$

7-C Complex Numbers

The square root of negative 1 is defined as $i = \sqrt{-1}$.

STANDARD FORM of a Complex Number is $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. Complex numbers are imaginary unless b is zero.

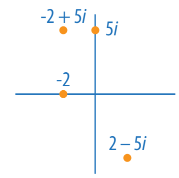
Imaginary numbers cannot be plotted on the real number line, but they can be plotted on the COMPLEX PLANE. In this plane, the number $a + bi$ is represented by the point (a, b).

❶ Plot numbers on the complex plane.

1. Write the number as $a + bi$. For real numbers, $b = 0$.
2. Plot the point (a, b).

❶ Plot the following numbers on the complex plane.

a) $2 - 5i$	b) $-2 + 5i$	c) -2	d) $5i$
1. $2 - 5i$	$-2 + 5i$	$-2 + 0i$	$0 + 5i$
$(2, -5)$	$(-2, 5)$	$(-2, 0)$	$(0, 5)$



The real parts of complex numbers are like terms, and the imaginary parts of complex numbers are like terms.

❷ Add and subtract complex numbers.

1. Distribute any coefficient, including negatives.
2. Find the total of the real parts, and find the total of the imaginary parts.

❷ $(3 + 5i) - 4(8 + 3i)$

1. $3 + 5i - 32 - 12i$
2. $-29 - 7i$

The square root of a negative number $-x$ can be written $\sqrt{-x} = \sqrt{x}\sqrt{-1} = \sqrt{x}i$. If $\sqrt{x}$ is not a whole number, the radical part is typically written last, after i.

❸ Simplify the square root of a negative integer.

1. Split the number up into a positive number times -1.
2. Simplify the square root of the positive number normally (see 7-B ❸), and the square root of -1, which is i.

❸ Siimplify.

a) $\sqrt{-25}$	b) $\sqrt{-75}$
$\sqrt{25}\sqrt{-1} = 5i$	$\sqrt{25}\sqrt{-1}\sqrt{3} = 5i\sqrt{3}$

Arithmetically, i functions the same as a variable like x, except that $i^2 = -1$.

❹ Multiply complex numbers.

1. Multiply normally, treating i as a variable.
2. Change each instance of i^2 to -1.

❹ Multiply.

a) $3i \cdot 4i$	b) $(2 + 3i)(5 - 6i)$	c) $(5 + 6i)(5 - 6i)$
1. $12i^2$	$10 + 15i - 12i - 18i^2$	$25 - 30i + 30i - 36i^2$
2. $-12(-1)$	$10 + 3i - 18(-1)$	$25 - 30i + 30i - 36(-1)$
-12	$10 + 3i + 18$	$25 + 36$
	$28 + 3i$	61

The COMPLEX CONJUGATE of a number $a + bi$ is $a - bi$. A number multiplied by its complex conjugate is a real number, as in the last example above.

The reciprocal of i is $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$.

❺ Divide by i.

1. Multiply by $-i$.

❺ $2 \div i$

1. $2(-i) = -2i$

7-D Completing the Square

If a perfect square trinomial is set equal to zero, it is easy to solve by first taking the square root of each side of the equation.

If the trinomial is not a perfect square, the same solving method can be used by COMPLETING THE SQUARE, which is adding the amount needed to make it a perfect square.

For $x^2 + bx + c$ to be a perfect square, c must equal the square of half of b: $c = \left(\frac{b}{2}\right)^2$. This is easiest if b is an even number.

❶ Solve $x^2 + bx + c = 0$ by completing the square.

1. Find half of b, and square that number to determine what value of c would make a perfect square.
2. Add or subtract whatever value is needed to make c equal to $\left(\frac{b}{2}\right)^2$, and add or subtract the same value from the other side of the equation as well. One way to do this is to subtract c from each side and then add $\left(\frac{b}{2}\right)^2$ to each side.
3. Factor the trinomial. It will factor as the perfect square $(x + \frac{b}{2})^2$.
4. Take the square root of each side.
5. Solve the + equation.
6. Solve the - equation.

❶ $x^2 + 10x + 9 = 0$

1. $c = (\frac{10}{2})^2 = 25$
2. $x^2 + 10x + 25 = -9 + 25$
3. $(x + 5)^2 = 16$
4. $x + 5 = \pm 4$
5. $x + 5 = 4$

   $x = -1$
6. $x + 5 = -4$

   $x = -9$

$ax^2 + bx + c = 0$ can also be solved by completing the square. This can be done by first dividing each term by a and then following the directions above, or by dividing $\left(\frac{b}{2}\right)^2$ by a when finding the value of c needed and then dividing every term by a once the correct value of c is reached. Either of these approaches will often lead to complicated fractions that make completing the square not an ideal method in these cases.

7-E The Quadratic Formula

The QUADRATIC FORMULA is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Once put in standard form $ax^2 + bx + c = 0$, any quadratic equation can be solved by the quadratic formula.

The radicand in the quadratic formula, $b^2 - 4ac$, is called the DISCRIMINANT.

❶ Solve a quadratic equation with the quadratic formula.

1. Put the equation in standard form.
2. Identify a, b, and c.
3. Calculate the discriminant $b^2 - 4ac$. Be sure to use parentheses when squaring or subtracting a negative number.
4. Calculate the square roots ($\pm$) of the discriminant.
5. Add the positive square root to -b, put this over $2a$, and simplify or round.
6. Repeat step 5 using the negative square root.

❶ $12x^2 + 17x = 7$

1. $12x^2 + 17x - 7 = 0$
2. $a = 12, b = 17, c = -7$
3. $b^2 - 4ac = 17^2 - 4(12)(-7) = 289 - (-336) = 625$
4. $\pm\sqrt{625} = \pm 25$
5. $x = \frac{-17 + 25}{24} = \frac{8}{24} = \frac{1}{3}$
6. $x = \frac{-17 - 25}{24} = \frac{-42}{24} = \frac{-7}{4}$

Alternatively, each solution to a quadratic equation can be calculated in a single step once a, b, and c are identified.

❶ $12x^2 + 17x = 7$

1. $12x^2 + 17x - 7 = 0$
2. $a = 12, b = 17, c = -7$
3. $x = \frac{-17 \pm \sqrt{17^2 - 4(12)(-7)}}{2(12)} \approx 0.33$ or -1.75

The discriminant determines what type of solutions a quadratic equation has.

❷ Use a discriminant to identify the type of solutions of a quadratic equation.

1. Put the equation in standard form $ax^2 + bx + c = 0$.
2. Evaluate the discriminant $b^2 - 4ac$.
3. If the discriminant is a square number, the solutions are rational (assuming a and b are rational).
   If the discriminant is a positive number but not square, the solutions are irrational (but still real).
   If the discriminant is negative, the solutions are imaginary.
   Because of the $\pm$ before the discriminant, there are two solutions to all quadratic equations except if the discriminant is zero.

❷ $2x^2 = 4x - 5$

1. $2x^2 - 4x + 5 = 0$
2. The discriminant is $(-4)^2 - 4(2)(5) = -24$
3. -24 is negative, so there are two imaginary solutions.

The words *solution*, *x-intercept*, *zero*, and *root* are sometimes used interchangeably, but they have slightly different uses.

ZEROS are values of the variable that make an expression equal to zero.

ROOTS are the same as zeros.

SOLUTIONS are values of the variable the make an equation true, and thus are the same as roots and zeros for equations set equal to zero.

X-INTERCEPTS are points on a graph where y is zero. Their x values are the same as the zeros, except they cannot be imaginary.

❸ Explain how zeros, roots, solutions, and x-intercepts are related for a given polynomial.

1. Set the polynomial equal to zero and solve.
2. Refer to the definitions above.

❸ $x^2 - 12x + 20$

1. $x^2 - 12x + 20 = 0$
   $(x - 2)(x - 10) = 0$
   $x = 2, x = 10$
2. The zeros and roots of $x^2 + 12x + 20$ are 2 and 10, because $x^2 - 12x + 20$ equals zero when $x = 2$ or $x = 10$.
   The solutions to $x^2 - 12x + 20 = 0$ are $x = 2$ and $x = 10$, because $x^2 - 12x + 20 = 0$ is true when $x = 2$ or $x = 10$.
   The x-intercepts of $y = x^2 - 12x + 20$ are (2, 0) and (10, 0), because $y = 0$ on the graph when $x = 2$ or $x = 10$.

There is a corresponding x-intercept for each real solution to $ax^2 + bx + c = 0$.

❹ Use a discriminant to identify the number of x-intercepts of a quadratic function.

1. Evaluate the discriminant $b^2 - 4ac$.
2. If the discriminant is positive, there are two x-intercepts.
   If the discriminant is zero, there is one x-intercept.
   If the discriminant is negative, there are no x-intercepts.

❹ $f(x) = 2x^2 - 4x + 5$

1. The discriminant is $(-4)^2 - 4(2)(5) = -24$
2. -24 is negative, so there are no x-intercepts.