Chapter Seven

Square Roots

Classification of Numbers

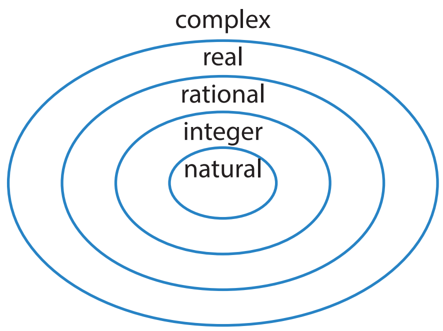
The solutions to many equations are not real numbers. This means they involve the square root of a negative number. In particular, the square root of -1 is defined as ***i***, which stands for *imaginary unit*.

Set	Symbol	Definition	Symbols	Examples
Natural	$\mathbb{N}$	counting numbers 1, 2, 3, . . .	none	4, 5, 8000
Integers	$\mathbb{Z}$	$\mathbb{N}$, -$\mathbb{N}$, and 0	-	all of $\mathbb{N}$, -4, -8000, 0
Rational	$\mathbb{Q}$	$\mathbb{Z} \div \mathbb{N}$	- . /	all of $\mathbb{Z}$, $-\frac{1}{2}$, 2.309, $\frac{2309}{1000}$
Real	$\mathbb{R}$	any number on the number line	any	all of $\mathbb{Q}$, $\sqrt{2}$, $\sqrt{3}$, π
Complex	$\mathbb{C}$	any number, including imaginary numbers	any	all of $\mathbb{R}$, $\sqrt{-2}$, $\sqrt{-3}$, $2+i$

Imaginary numbers are not real.

Irrational numbers are real but not rational.

Each number belongs to the set defining it and every set including that set, as shown at right. For example, all rational numbers are also real and complex.



Arithmetic with Square Roots

A **radicand** is an expression under a square root symbol.

A **radical** is a square root symbol and its radicand.

The **conjugate** of $a + \sqrt{b}$ is $a - \sqrt{b}$. $a + \sqrt{b}$ multiplied by its conjugate is the rational number $a^2 - b$, because the irrational terms will cancel.

Operation	Approach	Calculation	Example
Addition	Combine like terms.	$a\sqrt{x} + b\sqrt{x} = (a+b)\sqrt{x}$	$4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$
Multiplication	Multiply the radicands.	$\sqrt{x}\sqrt{y} = \sqrt{xy}$	$\sqrt{100}\sqrt{3} = \sqrt{300}$
Simplification	Split the radicand into two factors, one of which is a square.	$\sqrt{x^2y} = \sqrt{x^2}\sqrt{y} = x\sqrt{y}$	$\sqrt{300} = \sqrt{100}\sqrt{3} = 10\sqrt{3}$
Division	Multiply the numerator and denominator by the denominator's conjugate.	$\frac{x}{y+\sqrt{z}} \bullet \frac{y-\sqrt{z}}{y-\sqrt{z}} = \frac{xy - x\sqrt{z}}{y^2 - z}$	$\frac{10}{3+\sqrt{5}} \bullet \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{30-10\sqrt{5}}{3^2-5} = \frac{30-10\sqrt{5}}{4} = \frac{15-5\sqrt{5}}{2}$

Arithmetic with Complex Numers

The **complex conjugate** of $a + bi$ is $a - bi$. $a + bi$ multiplied by its complex conjugate is the real number $a^2 + b^2$, because the imaginary parts will cancel.

Operation	Approach	Calculation	Example
Addition	Add real parts, and add imaginary parts.	$(a + bi) + (c + di) =$ $(a + c) + (b + d)i$	$(3 + 5i) + (6 - 2i)$ $= (3 + 6) + (5 - 2)i$ $= 9 + 3i$
Multiplication	Treat i like a variable, but change any instance of i^2 to -1.	$ai \cdot bi = abi^2 = ab(-1)$ $= -ab$	$(3 + 2i)(1 - 4i)$ $= 3 - 12i + 2i - 8i^2$ $= 3 - 10i - 8(-1)$ $= 11 - 10i$
Simplification	Factor out $i = \sqrt{-1}$.	$\sqrt{-x} = \sqrt{-1}\sqrt{x} = i\sqrt{x}$	$\sqrt{-6}$ $= \sqrt{-1}\sqrt{6}$ $= i\sqrt{6}$
Division	Multiply the numerator and denominator by the denominator's conjugate.	$\frac{x}{y + zi} \cdot \frac{y - zi}{y - zi}$ $= \frac{xy - xzi}{y^2 - z^2i^2} = \frac{xy - xzi}{y^2 + z^2}$	$\frac{10}{3 + 5i} \cdot \frac{3 - 5i}{3 - 5i} = \frac{30 - 50i}{3^2 - 25i^2}$ $= \frac{30 - 50i}{34}$

Completing the Square

A perfect square trinomial can be written as a square, making it easy to take the square root.

For $x^2 + bx + c$ to be a square, c must be $\boldsymbol{c = (\frac{b}{2})^2}$. If it is not, the you can **complete the square** by adding (on each side) the amount needed to make it a square. The square will be $\boldsymbol{(x + \frac{b}{2})^2}$.

In the examples below with $b = 20$, c must be $(\frac{20}{2})^2 = 100$ to make it a square.

Equation	To make it a square	Work	Solutions
$\boldsymbol{x^2 + 20x + 100 = 0}$	This is already a square.	$x^2 + 20x + 100 = 0$ $(x + 10)^2 = 0$ $x + 10 = 0$	$x = -10$
$\boldsymbol{x^2 + 20x + 96 = 0}$	Add 4 to each side.	$x^2 + 20x + 100 = 4$ $(x + 10)^2 = 4$ $x + 10 = \pm 2$	$x = -12, x = -8$
$\boldsymbol{x^2 + 20x + 97 = 0}$	Add 3 to each side.	$x^2 + 20x + 100 = 3$ $(x + 10)^2 = 3$ $x + 10 = \pm\sqrt{3}$	$x = -10 \pm \sqrt{3}$
$\boldsymbol{x^2 + 20x + 104 = 0}$	Subtract 4 from each side.	$x^2 + 20x + 100 = -4$ $(x + 10)^2 = -4$ $x + 10 = \pm 2i$	$x = -10 \pm 2i$

The Quadratic Formula

Solving the equation $ax^2 + bx + c = 0$ by completing the square yields the solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is called the **quadratic formula**, and it can be used to solve any quadratic equation in standard form.

The radicand $b^2 - 4ac$ is called the **discriminant**. Be sure to use parentheses around negative numbers.

Equation	Discriminant	Equation	Solutions
$x^2 + 8x + 16 = 0$	$8^2 - 4(1)(16) = 0$	$x = \frac{-8 \pm \sqrt{0}}{2(1)}$	$x = \frac{-8 \pm 0}{2} = -4$
$x^2 + 8x + 12 = 0$	$8^2 - 4(1)(12) = 16$	$x = \frac{-8 \pm \sqrt{16}}{2(1)}$	$x = \frac{-8 \pm 4}{2} = -2, -6$
$x^2 - 8x - 12 = 0$	$(-8)^2 - 4(1)(-12) = 112$	$x = \frac{8 \pm \sqrt{112}}{2(1)}$	$x = \frac{8 \pm 4\sqrt{7}}{2} = 4 \pm 2\sqrt{7}$
$x^2 - 8x + 20 = 0$	$(-8)^2 - 4(1)(20) = -16$	$x = \frac{8 \pm \sqrt{-16}}{2(1)}$	$x = \frac{8 \pm 4i}{2} = 4 \pm 2i$

The discriminant indicates whether the solutions will be real or imaginary. Since imaginary x-intercepts don't exist, it also indicates how many x-intercepts the graph has.

Discriminant	Solutions	# of x-intercepts	Factorable
positive	two real	two	if discriminant is square
zero	one real	one	as a perfect square
negative	two nonreal	none	no

Zeros, Roots, Solutions, and *X*-Intercepts

The terms **zero**, **root**, **solution**, and ***x*-intercept** are sometimes used interchangably, but they have slightly different uses.

Term	Definition	Example
Zero	value that makes the expression equal to zero	The zeros of the expression $x^2 - 7x + 10$ are 2 and 5.
Root	same as a zero	The roots of the expression $x^2 - 7x + 10$ are 2 and 5.
Solution	value that makes the equation true	The solutions of the equation $x^2 - 7x + 10 = 0$ are $x = 2$ and $x = 5$.
***X*-Intercept**	point on the graph where y is zero	The x-intercepts of the graph of $y = x^2 - 7x + 10$ are $(2, 0)$ and $(5, 0)$.